Recurrences
Since recurrences are a major tool in the analysis of algorithms, this assignment deals with them exclusively.

1. Consider the first order linear recurrence

\[ a_{n+1} = 2a_n + \frac{1}{n+1}, \quad n \geq 0, \quad a_0 = 0. \]

(a) Present the solution of the recurrence as a sum (you are not expected to find a closed form for the sum. Indeed — no such form seems to exist).

(b) This is a part you need only to do if you want a bonus grade: for large \( n \) there is a very good approximation for the value of \( a_n \). What is it?

*Hint:* Use the fact that \( \int_0^x x^j dx = a^{j+1}/(j+1) \).

2. Consider the following recurrence

\[ x_n = x_{n-1} + ax_{n-2} + bn, \quad n \geq 2, \]

(a) Solve it in terms of the constants \( a \) and \( b \), and the initial values \( x_0 \) and \( x_1 \). Assume whatever you need about all these constants to keep the solution simple.

(b) Now show one or more particular situations where values of the constants can make the above solution incorrect. Solve the recurrence for those special cases.

3. Finally, solve the following recurrence

\[ y_{n+3} = 4y_{n+2} - y_{n+1} - 6y_n + 2^n, \quad n \geq 0, \quad y_j = j, \quad 0 \leq j \leq 2. \]

There is nothing peculiar about this equation, except that when you consider the characteristic equation you may realize you never actually solved a cubic equation. Here all the roots can be guessed with a little daring, but let me tell you that \(-1\) is one of them, and once you have a single root you can reduce the cubic to a quadratic, with which you are familiar.

The idea is that if \( a \) is the root of a polynomial equation \( P(x) = 0 \), this means \( P(x) \) has the factor \( x - a \). If you divide \( P(x) \) by this factor you are left with a polynomials of one degree lower, \( P'(x) = P(x)/(x-a) \), and the equation \( P'(x) = 0 \) is easier, and will yield the rest of the roots.