1. We use summation factors to derive $x_n = \left( \prod_{n \geq k \geq 1} \frac{k+3}{k+1} \right) \left( 0 + \sum_{n \geq k \geq 1} \frac{k+3}{j+1} \right)$. Since

$$
\prod_{n \geq k \geq 1} \frac{k+3}{k+1} = \frac{(n+3)(n+2)}{2},
$$

then $x_n = \frac{(n+3)(n+2)}{2} \sum_{n \geq k \geq 1} \frac{6}{k+2}$. Substituting $k-2$ for $k$,

$$
x_n = (n+3)(n+2) \sum_{n \geq k \geq 3} \frac{1}{k} = (n+3)(n+2) \left( \sum_{n \geq k \geq 3} \frac{1}{k} \right) = (n+3)(n+2) \left( H_{n+2} - \frac{3}{2} \right).
$$

2. A) $p_n = \begin{cases} 
1, & \text{if } n = 0 \\
0, & \text{if } n = b \\
\frac{1}{2} p_{n+1} + \frac{1}{2} p_{n-1}, & \text{if } 0 < n < b
\end{cases}
$

B) To solve the second-order homogeneous recurrence $p_{n+1} = 2p_n - p_{n-1}$, we use the characteristic equation $x^2 - 2x + 1 = 0$ which has the characteristic root $x = 1$ of multiplicity 2. Hence, our solution is of the form $p_n = \alpha 1^n + \beta n 1^n = \alpha + \beta n$. When the boundary conditions are substituted, this implies that the solution is $p_n = 1 - \frac{n}{b}$.

3. (A) Letting $t_h$ denote the maximum number of nodes in a Fascist tree of hauteur $h$, it follows that the children of the root are maximal Fascist trees of hauteur $h-1$ and $h-2$. Hence,

$$
t_h = \begin{cases} 
1, & \text{if } h = 1 \\
2, & \text{if } h = 2 \\
1 + t_{h-1} + t_{h-2}, & \text{if } h > 2
\end{cases}
$$

(B) Using the characteristic equation $(x^2 - x - 1)(x-1) = 0$, we find the characteristic roots $x_1 = \frac{1 + \sqrt{5}}{2}, x_2 = \frac{11 \sqrt{5}}{2}, x_3 = 1$. The solution to the recurrence is of the form $t_h = \alpha \left( \frac{1 + \sqrt{5}}{2} \right)^h + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^h + \gamma 1^h$. Plugging in initial conditions yields

$$
t_h = \left( 1 + \frac{2}{\sqrt{5}} \right) \left( \frac{1 + \sqrt{5}}{2} \right)^h + \left( 1 - \frac{2}{\sqrt{5}} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^h - 1.
$$