C.S.504
Solution for H.W. #5

1) a. \(a=2,\ b=2,\ f(n)=n^3,\ n^{\log_b a} = n^{\lg 2} = n,\ f(n) = n^3\) and case 3 of the Master Theorem applies, with \(T(n) = \Theta(n^3)\).

b. \(a=1,\ b=10/9,\ f(n)=n,\ n^{\log_b a} = n^{\log_{10} 1} = 1,\ f(n) = n\) and case 3 of the Master Theorem applies, with \(T(n) = \Theta(n)\).

c. \(a=16,\ b=4,\ f(n)=n^2,\ n^{\log_b a} = n^{\log_4 16} = n^2,\ f(n) = n^2\) and case 2 of the Master Theorem applies, with \(T(n) = \Theta(n^2 \lg n)\).

d. \(a=7,\ b=3,\ f(n)=n^2,\ n^{\log_b a} = n^{\log_3 7},\ f(n) = n^2\) and case 3 of the Master Theorem applies, with \(T(n) = \Theta(n^2 \cdot 2^{-\log_3 7})\).

e. \(a=7,\ b=2,\ f(n)=n^2,\ n^{\log_b a} = n^{\log_2 7},\ f(n) = n^2\) and case 1 of the Master Theorem applies, with \(T(n) = \Theta(n^2 \cdot \lg 7)\).

f. \(a=2,\ b=4,\ f(n) = \sqrt{n},\ n^{\log_b a} = n^{\log_4 2} = \sqrt{n},\ f(n) = \sqrt{n}\) and case 2 of the Master Theorem applies, with \(T(n) = \Theta(\sqrt{n \lg n})\).

2) \(t_n = n + 42 \sum_{k=0}^{n-1} t_k\)

\(t_{n+1} = n + 1 + 42 \sum_{k=0}^{n} t_k = n + 1 + 42 t_n + 42 \sum_{k=0}^{n-1} t_k\)

Subtracting the top line from the bottom, \(t_{n+1} - t_n = 42 t_n + 1\), yielding the first-order linear nonhomogeneous recurrence \(t_{n+1} = 43 t_n + 1\), which has the solution (via summation factors or characteristic equations)

\(t_n = \frac{1}{42} (43^n - 1)\).

3). \(G(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1 - 26z}\).

4) The \(k^{th}\) integer in the composition is \(z + z^2 + z^3 + \ldots = \frac{z}{1 - z}\), although since no term can be greater than \(n\), this could also be written as \(z + z^2 + z^3 + \ldots + z^n = \frac{z - z^{n+1}}{1 - z}\).

Allowing any number of parts in the composition yields the generating function

\(G(z) = \prod_{k=1}^{\infty} \frac{z}{1 - z} = \prod_{k=0}^{\infty} \frac{z}{1 - z} - 1 = \frac{1 - z}{1 - 2z}\).