C.S.504
Solution for H.W. #4

1) If the queue is empty 50% of the time, then \( r = 0.5 \), and the expected queue length is 1. If the queue is empty 10% of the time, then \( r = 0.1 \), and the expected queue length is 9.

2) For \( n > 0 \),
\[
(n + 1)a_{n+1} = (n + 2)a_n + 4(n + 1)^2
\]
\[
a_{n+1} = \frac{n + 2}{n + 1}a_n + 4(n + 1)
\]
Using summation factors, \( b_n = \frac{n + 1}{n} \) and \( c_n = 4n \). Noting that
\[
\prod_{k=1}^{n} b_k = \frac{n + 1}{n} \cdots \frac{2}{1} = n + 1,
\]
we find
\[
a_n = b_n \cdots b_1 \left( a_0 + \sum_{1 \leq k \leq n} \frac{c_k}{b_k \cdots b_1} \right) = (n + 1) \sum_{1 \leq k \leq n} \frac{4k}{k + 1}.
\]
Pulling the 4 outside the summation and replacing \( k \) by \( k - 1 \),
\[
a_n = 4(n + 1) \sum_{2 \leq k \leq n + 1} \frac{k - 1}{k} = 4(n + 1) \left( \sum_{2 \leq k \leq n + 1} 1 - \sum_{2 \leq k \leq n + 1} \frac{1}{k} \right)
\]
\[
= 4(n + 1) \left( n + \sum_{1 \leq k \leq n + 1} \frac{1}{k+1} \right)
\]
\[
= 4(n + 1) \left( n + 1 - H_{n+1} \right) \quad \text{for } n > 0.
\]

3. A) A worst-case graph eliminates as few vertices as possible each pass. Since at least 1 vertex belongs to \( \Gamma(v) \), a worst case graph has \( n \) vertices \( \{v_1, \ldots, v_n\} \) and \( n/2 \) edges \( \{(v_1, v_2), (v_3, v_4), \ldots, (v_{n-1}, v_n)\} \).

B) Letting \( T(n) \) denote the worst-case execution time of \( A(G) \) on a graph of \( n \) vertices,
\[
T(n) = \begin{cases} 1, & \text{if } n = 1 \\ T(n - 1) + T(n - 2) + cn^2, & \text{if } n > 1 \end{cases}
\]

C) The characteristic equation of the above recurrence is \((x^2 - x - 1)(x - 1)^3 = 0\), and the solution is of the form \( T(n) = \alpha \left( \frac{1 + \sqrt{5}}{2} \right)^n + \beta \left( \frac{1 - \sqrt{5}}{2} \right)^n + \gamma n^2 + \delta n + \epsilon n^2 = \Theta(1.619^n) \)