C.S.504  
Solution for H.W. #2

1) A) (1,5), (2,5), (3,4), (3,5), (4,5)

B) \( \sum_{j=1}^{n} (j-1) = \sum_{j=1}^{n} j - \sum_{j=1}^{n} 1 = \frac{n(n+1)}{2} - n = \frac{n(n-1)}{2} \) inversions.

C) The do-loop is executed exactly once for each inversion. The execution time of INSERTION-SORT has a linear factor plus a factor proportional to the number of inversions in the input array.

D) \( X_j, 1 \leq j \leq n, \) - # inversions \( (j,i) \) for each \( j \) counting from the right, \( n-j \), (that is, for \( j=1 \) the only possible inversion is \( (n-1,n) \))

\[ X = \sum_{n \geq j \geq 1} X_j. \]

We want \( E[X] \). For each \( j, X_j \) can assume values \( 0, \ldots, j \), Assuming all permutations of \( A \) equally likely (drawn from uniform distribution), \( \Pr\{X_j = i\} = \frac{1}{j+1}, 0 \leq i \leq j \), and

\[ E[X_j] = \sum_{j=1}^{n} \frac{i}{j+1} = \frac{1}{j+1} \sum_{j=2}^{i} \frac{1}{j} = \frac{i}{2}. \]

Now, \( E[X] = \sum_{j=1}^{n} E[X_j] = \sum_{j=1}^{n} \frac{i}{2} = \frac{1}{2} \sum_{j=1}^{n} - \frac{n(n+1)}{2} = \frac{n(n-1)}{4} = \Theta(n^2) \)

E) decrease

2. Letting \( \text{rv} A_n \) denote the number of accesses,

\[ VA_n = E[A_n^2] - \left( E[A_n] \right)^2 = \sum_{1 \leq k \leq n} k^2 \Pr\{A_n = k\} - \left( \sum_{1 \leq k \leq n} k \right)^2 = \sum_{1 \leq k \leq n} k^2 \omega_k^2 - \left( \sum_{1 \leq k \leq n} k \right)^2 = \sum_{1 \leq k \leq n} k \left( n - \omega_k \right)^2 \]

\[ = \frac{n(n+1)}{2} \left( \frac{1}{2} - \omega_k \right) + \frac{1}{2}, \]

and the standard deviation of \( A_n \) is

\[ \sqrt{\frac{n}{2} \left( \frac{1}{2} - \omega_k \right) + \frac{1}{2}}. \]

3. There are \( \frac{n+1}{2} \), leaves, all at height 0, and \( \frac{n+1}{4} \) nodes of height 1,\ldots. Letting \( T(n) \) denote the sum of the heights,

\[ T(n) = \left( \frac{n+1}{2} \right) 0 + \left( \frac{n+1}{4} \right) 1 + \left( \frac{n+1}{8} \right) 2 + \ldots + \left( \frac{n+1}{n+1} \right) \left( \frac{n+1}{n+1} \right) = (n+1) \sum_{k=1}^{\frac{1}{2}(n+1)} k \cdot \frac{1}{2^k}. \]

Substituting \( m = \log(n+1) \) and solving the first inner sum,

\[ \sum_{k=1}^{m} \frac{1}{2^k} = \sum_{k=0}^{m} \frac{1}{2^k} = \left( \frac{1}{2} - \left( \frac{1}{2} \right)^{m+1} \right) + \left( \frac{1}{2} \right)^{m+1} = 4 \left( \frac{1}{2} - \left( \frac{1}{2} \right)^{m+1} \right) + \left( \frac{1}{2} \right)^{m+1} = 2 - 2 \left( \frac{1}{2} \right)^{m+1} \]

Solving the second inner sum,

\[ \sum_{k=1}^{m} \frac{1}{2^k} - 1 = \frac{1 - \left( \frac{1}{2} \right)^{m+1}}{1 - \frac{1}{2}} = 2 - \left( \frac{1}{2} \right)^{m+1} - 1 = \frac{1}{n+1}. \]

Recombining,

\[ T(n) = (n+1) \left( 2 - \frac{2}{n+1} \left( \frac{1}{2n+1} + 1 \right) - 1 + \frac{1}{n+1} \right) = n - \log(n+1). \]