1. (5 points) We define an *Isaac tree* $I_n$ recursively by:
- $I_0$ consists of a single node,
- the *Isaac tree* $I_n$, $n \geq 1$, consists of two Isaac trees $I_{n-1}$ such that the root of one is the rightmost child of the root of the other.

The first four Isaac trees are:

A) How many nodes are in $I_n$?
B) What is the height of $I_n$?
C) Defining the *depth* of a node in a tree to be the number of edges on the simple path from the root to the node, how many nodes are at depth $k$ of *Isaac tree* $I_n$, for $0 \leq k \leq n$? Justify your response. Examining the above figure shows that $I_3$ has 3 nodes at depths 1 and 2, and 0 nodes at depths 0 and 3.

2) (3 points) From Cormen, Leiserson & Rivest's *Algorithms*: Show that for any integers $n, j, k \geq 0, j+k \leq n$, \[
\binom{n}{j+k} \leq \binom{n}{j} \binom{n-j}{k}.
\] Give values for $n, j$ and $k$ such that equality does not hold.

3) (5 points) Give a closed form (which may involve binomial coefficients, but should not involve $\Sigma$) for \[
\sum_{0 \leq j \leq n} \binom{j}{k}.
\] Note that for $n=4$ and $k=2$, the value is 35.

(*Hint*: It is helpful to manipulate things so that you can get rid of the $j$ multiplier, although this may create more than one sum.)
C.S.504
Solution for H.W. #3

1) A) By induction, we show that \( I_n \) has \( 2^n \) nodes. Clearly \( I_0 \) has \( 2^0=1 \) node.
   Assuming that \( I_{n-1} \) has \( 2^{n-1} \) nodes, we see that \( I_n \) has \( 2^{n-1}+2^{n-1}=2^n \) nodes.
   B) Since the height of \( I_0 \) is 0 and the height of \( I_n \) is one greater than the
   height of \( I_{n-1} \), then the height of \( I_n \) is \( n \).
   C) Letting \( d(n,k) \) denote the number of nodes at depth \( k \) of \( I_n \), we prove by
   induction (on \( k \)) that \( d(n,k) = \binom{n}{k} \) by noting that \( d(n,0) = 1 = \binom{n}{0} \) for \( n \geq 0 \). Since
   the nodes at depth \( k \) are the nodes at depth \( k \) in one of the trees \( I_{n-1} \) plus the
   nodes at depth \( k-1 \) in one of the trees \( I_{n-1} \), we see that
   \[
   d(n,k) = d(n-1,k) + d(n-1,k-1) = \binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.
   \]

2) \[
\binom{n}{j} \binom{n-j}{k} = \frac{n!}{j!(n-j)!} \frac{(n-j)!}{k!(n-j-k)!} = \frac{n!}{(n-J+k)!(n-J-k)!} = \frac{n!}{(n-J+k)!(n-J-k)!} \cdot \binom{n-J+k}{j}
\]
   which is greater than or equal to \( \binom{n}{j+k} \) since \( k \geq 0 \) implies
   \[
   \binom{j+k}{j} \geq 1.
   \]
   Equality holds when \( k=0 \) or \( j=0 \). Choosing \( n=2, j=k=1 \) we have an
   inequality since \( \frac{2}{2} = 1 \leq \frac{2}{1} \cdot \frac{1}{1} = 2 \)

3) \[
\sum_{0 \leq j \leq n} \frac{j}{k} = \sum_{0 \leq j \leq n} \frac{j}{k+1} = \sum_{0 \leq j \leq n} \left( \frac{j+1}{k+1} - \frac{j}{k+1} \right) = \sum_{0 \leq j \leq n} \left( \frac{j+1}{k+1} \right) - \sum_{0 \leq j \leq n} \left( \frac{j}{k+1} \right) = (k+1) \sum_{0 \leq j \leq n} \left( \frac{j}{k+1} \right) - (n+1) \sum_{0 \leq j \leq n} \left( \frac{j+1}{k+1} \right) = (k+1) \sum_{0 \leq j \leq n} \left( \frac{j}{k+1} \right) - (n+1) \sum_{0 \leq j \leq n+1} \left( \frac{j}{k+1} \right) = (k+1) \sum_{0 \leq j \leq n+1} \left( \frac{j}{k+1} \right) - (n+1) \sum_{0 \leq j \leq n+1} \left( \frac{j+1}{k+1} \right) = (k+1) \sum_{0 \leq j \leq n+1} \left( \frac{j}{k+1} \right) - (n+1) \sum_{0 \leq j \leq n+1} \left( \frac{j+1}{k+1} \right)
\]