1. (5 points) A simple path in a graph is a sequence of adjacent edges such that no vertex is visited more than once. The length of a path is the number of edges on the path.

A) Run and test a program to estimate the number of simple paths of length $k$, $0 \leq k \leq 19$, for the following graph. There are 20 simple paths of length 0.

B) Compute how the estimate and variance of the estimate changes with the amount of work in computing the estimate.
2. (2 points) Find a closed form for \( \sum_{0 \leq k \leq n} k^2 x^k \).

3. (4 points) Define the height of a node in a rooted tree as the maximum number of edges on a path from the node to a leaf beneath the node. Assume there is a positive integer \( m \) such that \( n = 2^m \), and consider a full binary tree with \( n-1 \) nodes and with all leaves at the same distance from the root. For example, find a closed form for the expression \( \sum_{\eta} \text{height}(\eta) \), where the sum is taken over all nodes \( \eta \). For example, for the tree above this value is 4. (Hint: You may want to evaluate \( (# \text{ nodes at height \geq 1}) + (# \text{ nodes at height \geq 2}) + \ldots \) )
2) Setting \( S_n = \sum_{0 \leq k \leq n} k^2 x^k \), we use the perturbation method and solve

\[
S_{n+1} = \sum_{0 \leq k \leq n+1} k^2 x^k = (n+1)^2 x^{n+1} + S_n.
\]

Replacing \( k \) by \( k+1 \) in the leftside of the final equality yields

\[
\sum_{1 \leq k \leq n+1} (k+1)^2 x^{k+1} = \sum_{0 \leq k \leq n} (k^2 + 2k + 1) x^k = x \sum_{0 \leq k \leq n} k^2 x^k + 2x \sum_{0 \leq k \leq n} k x^k + x \sum_{0 \leq k \leq n} x^k
\]

\[
= xS_n + 2x \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} + x \frac{1 - x^{n+1}}{1-x}.
\]

Setting this equal to the rightside of the perturbation equality yields

\[
(n+1)^2 x^{n+1} + S_n = xS_n + 2x \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} + x \frac{1 - x^{n+1}}{1-x}.
\]

A bit more algebra yields

\[
S_n(1-x) = 2x \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} + x \frac{1 - x^{n+1}}{1-x} - (n+1)^2 x^{n+1}.
\]

\[
S_n = 2x \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^3} + x \frac{1 - x^{n+1}}{(1-x)} - (n+1)^2 x^{n+1}
\]

\[
= \frac{2x^2 - 2(n+1)x^{n+2} + 2nx^{n+3} + x(1-x)(1-x^n) - (1-x)^2(n+1)^2 x^{n+1}}{(1-x)^3}
\]

\[
= \frac{2x^2 - 2(n+1)x^{n+2} + 2nx^{n+3} + x - x^{n+2} - x^2 + x^{n+3} - (n+1)^2(x^{n+1} - 2x^{n+2} + x^{n+3})}{(1-x)^3}
\]

\[
= \frac{x^2 + x - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}
\]

3) Using the hint yields

\[
\left(\frac{n}{4} + \frac{n}{8} + \ldots + 1\right) + \left(\frac{n}{8} + \frac{n}{16} + \ldots + 1\right) + \ldots + 1
\]

where there are \( m-1 = \lg n - 1 \) terms. Recognizing that each of these series are geometric series, they may be rewritten as

\[
\left(\frac{n}{2} - 1\right) + \left(\frac{n}{4} - 1\right) + \ldots + 1 = \left(\frac{n}{2} + \frac{n}{4} + \ldots + 2\right) - (\lg n - 1) = n - 2 - (\lg n - 1) = n - 1 - \lg n
\]