1. (2 points) What does the following algorithm compute?
   
   ```plaintext
   function f(n, m : integer) : integer;
   \( X \leftarrow m; \)
   \( K \leftarrow 0; \)
   \textbf{while} \( X \leq n \) \textbf{do}
   \( X \leftarrow X \times m; \)
   \( K \leftarrow K + 1; \)
   \textbf{return}(K);
   ```

2. (9 points) We want to find the maximum and the minimum elements of an array \( L[1..6] \) which contains some permutation of the integers \( (1, 2, 3, 4, 5, 6) \). The work performed by an algorithm will be the number of pairwise comparisons of integers. Consider the following two algorithms:

   **Algorithm P**
   ```plaintext
   max \leftarrow L[1]
   min \leftarrow L[1]
   \textbf{for} \( k \leftarrow 2 \textbf{ to } 6 \) \textbf{do}
   \textbf{if} \( L[k] > \text{max} \) \textbf{then} \( \text{max} \leftarrow L[k] \)
   \textbf{else if} \( L[k] < \text{min} \) \textbf{then} \( \text{min} \leftarrow L[k] \)
   ```

   **Algorithm Q**
   ```plaintext
   \textbf{if} \( L[1] < L[2] \) \textbf{then} \( \text{maxes}[1] \leftarrow L[2]; \text{mins}[1] \leftarrow L[1] \)
   \textbf{else} \( \text{mins}[1] \leftarrow L[2]; \text{maxes}[1] \leftarrow L[1] \)
   \textbf{else} \( \text{mins}[2] \leftarrow L[4]; \text{maxes}[2] \leftarrow L[3] \)
   \textbf{else} \( \text{mins}[3] \leftarrow L[6]; \text{maxes}[3] \leftarrow L[5] \)
   
   \( \text{max} \leftarrow \text{maxes}[1] \)
   \( \text{min} \leftarrow \text{mins}[1] \)
   \textbf{for} \( k \leftarrow 2 \textbf{ to } 3 \) \textbf{do}
   \textbf{if} \( \text{maxes}[k] > \text{max} \) \textbf{then} \( \text{max} \leftarrow \text{maxes}[k] \)
   \textbf{if} \( \text{mins}[k] < \text{min} \) \textbf{then} \( \text{min} \leftarrow \text{mins}[k] \)
   ```
We assume that the each permutation of (1, 2, 3, 4, 5, 6) is equally likely. For your homework, you will establish a probability model for the number of pairwise comparisons for each algorithm for this input data.

A) Define a sample space.
B) For algorithm P, define a sequence of ten random variables, five denoting how many times the comparison \( L[k] > \text{max}, \ 2 \leq k \leq 6 \), will be performed, and five denoting how many times the comparison \( L[k] < \text{min}, \ 2 \leq k \leq 6 \), will be performed. For example, for input (2, 4, 6, 1, 3, 5), the comparison \( L[5] > \text{max} \) will be performed one time and the comparison \( L[5] < \text{min} \) will be performed one time.
C) What is the expected value of each of the ten random variables of question B)?
D) What is the expected number of binary comparisons for each of algorithms P and Q?
E) What is the variance of the number of comparisons of algorithms P and Q?
F) What is the standard deviation of the number of comparisons of algorithms P and Q?
G) What is the probability that ten comparisons will be performed in algorithm P?
H) What bound does Chebyshev's inequality provide on the probability that ten comparisons will be performed in algorithm P? Compare this to the bound of part G).
C.S.504
Solution for H.W. #1

1) \( f(n,m) = \lfloor \log_m n \rfloor \)

2) A) \( \Omega \) is the set of 6! permutations of \((1, 2, 3, 4, 5, 6)\)

B) \( X_k = \begin{cases} \text{execute if} & k \in \Pi(k) \geq \max_{2 \leq k \leq 6} \text{execute otherwise} \\ \text{0 otherwise} \end{cases} \)
and \( Y_k = \begin{cases} \text{execute if} & k \in \Pi(k) < \min_{2 \leq k \leq 6} \text{execute otherwise} \\ \text{0 otherwise} \end{cases} \)

C) \( E[X_k] = 1 \) and \( E[Y_k] = 0 \)* \( \frac{1}{k} + \frac{k-1}{k} = \frac{k-1}{k} \) for \( 2 \leq k \leq 6 \).

D) For algorithm \( P \), we define \( X = \sum_{2 \leq k \leq 6} X_k + \sum_{2 \leq k \leq 6} Y_k \) and

\[
E[X] = \sum_{2 \leq k \leq 6} E[X_k] + \sum_{2 \leq k \leq 6} E[Y_k] = 5 + \sum_{2 \leq k \leq 6} \frac{k-1}{k} = 8.55.
\]
For algorithm \( Q \), the expected number of comparisons is 7.

E For algorithm \( P \), since the 10 r.v.s are independent,

\[
V \left( \sum_{2 \leq k \leq 6} X_k + \sum_{2 \leq k \leq 6} Y_k \right) = \sum_{2 \leq k \leq 6} V X_k + \sum_{2 \leq k \leq 6} V Y_k.
\]
For each \( X_k, 2 \leq k \leq 6 \),

\[
V X_k = E \left[ (X_k - E[X_k])^2 \right] = 0.
\]
For each \( Y_k, 2 \leq k \leq 6 \),

\[
V Y_k = E \left[ (Y_k - E[Y_k])^2 \right] = \frac{k-1}{k} \left( 1 - \frac{k-1}{k} \right)^2 + \frac{1}{k} \left( 0 - \frac{k-1}{k} \right)^2 = \frac{k-1}{k^2}.
\]
So for algorithm \( P \), the variance is

\[
V \left( \sum_{2 \leq k \leq 6} X_k + \sum_{2 \leq k \leq 6} Y_k \right) = \sum_{2 \leq k \leq 6} \frac{k-1}{k^2} = 0.9586111111111111.
\]
For algorithm \( Q \), the variance of the number of comparisons is 0.

F For algorithm \( P \), the standard deviation of the number of comparisons is \( \sqrt{0.9586111111111111} = 0.9790868761816344 \). For algorithm \( Q \), the standard deviation of the number of comparisons is 0.

G) Ten comparisons are performed if the list is sorted in increasing order. Since every permutation of \( L \) is equally likely, the probability of this event is \( \frac{1}{6} = 0.166666667 \).

H) Ten comparisons is \( \frac{10 - 8.55}{0.9790868} = 1.4809718607175577 \) standard deviations from the mean of 8.55 comparisons. According to Chebyshev's inequality, the probability of this event is

\[
\Pr \left[ |X - E[X]| \geq 1.48097 \right] \leq \frac{1}{1.48097^2} = 0.4559386263658696
\]