1. (4 points) Solve the recurrence \( t_n = \begin{cases} 5t_{n-1} - 6t_{n-2} + 4 \cdot 3^n & \text{if } n > 1 \\ 36 & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases} \)

2. (4 points) Solve the recurrence \( t_n = \begin{cases} \frac{2t_n}{n} + n\log n & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases} \)

if \( n > 0 \) is a power of 2.

3. (4 points) One version of QuickSort has a best-case recurrence for its execution time of \( t_n = \begin{cases} \frac{2t_n}{n} + an + b & \text{if } n > 1 \\ c & \text{if } n = 1 \end{cases} \)

for constants \( a, b \) and \( c \).

Find an exact solution for the recurrence.
1. The characteristic equation is \((x^2 - 5x + 6)(x - 3)^2 = 0\), which has characteristic roots 2 of multiplicity 1 and 3 of multiplicity 2. The solution of the recurrence is of the form \(t_n = a_2n + b_3n^2 + gn^3\), with the constants chosen to satisfy initial conditions \(t_0 = 0 = a + b\), \(t_1 = 36 = 2a + 3b + 3g\), \(t_2 = 5t_1 - 6t_0 + 4 \cdot 3^2 = 216 = 4a + 9b + 18g\) which yields \(a = 0, b = 0\) and \(g = 12\), and \(t_n = 12n^3\).

2. Replace \(n\) by \(2^k (k = \lg n)\), so that \(t_{2^k} = 2a_{2^k} + 2^k \lg 2^k = 2t_{2^k-1} + k2^k\).

Let \(T_k = t_{2^k}\) so that \(T_k = 2T_{k-1} + k2^k\), which has characteristic equation \((x - 2)^3 = 0\), which has characteristic root 2 of multiplicity 3. The solution is \(T_k = t_{2^k} = t_n = a_2^k + b_3^k + \gamma n^2\) which yields \(t_1 = 1 = \alpha, t_2 = 4 = 2t_1 + 2\lg 2 = 2\alpha + 2\beta + 2\gamma,\) and \(t_4 = 2t_2 + 4\lg 4 = 16\), and finally \(t_n = n(1 + \frac{\lg n}{2} + \frac{\lg^2 n}{2})\).

3. Replace \(n\) by \(2^k (k = \lg n)\), so that \(t_{2^k} = 2t_{2^k} + a_2^k + b = 2t_{2^k-1} + a2^k + b\). Let \(T_k = T_{2^k}\). \(T_0 = t_1 = c, T_k = 2T_{k-1} + a2^k + b\). Using summing factors, with \(b_k = 2\) and \(c_k = a2^k + b\), we get

\[
T_k = 2^k \left( c + \sum_{1 \leq j \leq k} \frac{a_j^k + b}{2^j} \right) = c2^k + 2^k \left( \sum_{1 \leq j \leq k} a + b \sum_{1 \leq j \leq k} \left( \frac{1}{2} \right)^j \right)
\]

\[
= c2^k + ka2^k + b2^k \left( \frac{1 - \left( \frac{1}{2} \right)^{k+1}}{1 - \frac{1}{2}} - 1 \right) = 2^k(c + ka) + b2^k \left( 2 - \left( \frac{1}{2} \right)^k - 1 \right)
\]

\[
= 2^k(c + ka) + b2^k - b. \quad \text{Substituting back yields}
\]

\[
t_n = t_{2^k} = T_k = 2 \lg n(c + \lg n) + b2^\lg n - b = cn + an \lg n + bn - b = an \lg n + (b + c)n - b
\]