1. (3 points) Use the method of characteristic roots to solve the recurrence:
\[ t_n = t_{n-2} + 4n, \quad n \geq 2 \]
\[ t_0 = 1, \quad t_1 = 4 \]

2. (3 points) Use the method of characteristic roots to solve the recurrence:
\[ t_n = 2t_{n-1} + (n + 5)3^n, \quad n \geq 1 \]
\[ t_0 = 0 \]

3. (5 points) (From Cormen, Leiserson & Rivest) An array \( A[1..n] \) contains all the integers from 0 to \( n \) except one. The elements of \( A \) are represented in binary, and the only operation we can use to access them is of the form "fetch the \( j \)th bit of \( A[i] \)", which takes constant time. Assume each entry of \( A \) is expressed as a sequence \( b_{\lfloor \lg(n+1) \rfloor}, \ldots, b_1 \) of bits. Show that there exists a \( k \) such that the missing integer can be found using only \( kn \) bit fetches.
Note that storing \( A[1..n] \) requires \( n\lfloor \lg(n+1) \rfloor \) bits, so the solution can not involve fetching every bit of \( A \).
1. The characteristic equation is \((x^2 - 1)(x - 1)^2 = (x - 1)^3(x + 1) = 0\)

The solution is of the form

\[ t_n = a_1n + b_1n + g_1n^2 + d(-1)^n = a + b + g + d(-1)^n \]

Plugging in to initial conditions yields

\[ t_0 = 1 = a + d \]
\[ t_1 = 4 = a + b + g - d \]
\[ t_2 = 9 = a + 2b + 4g + d \]
\[ t_3 = 16 = a + 3b + 9g - d \]

Solving this system of equations yields \(a = 1, b = 2, g = 1, d = 0\)

Therefore, \(t_n = 1 + 2n + n^2 = (n + 1)^2\)

2. The characteristic equation is \((x - 2)(x - 3)^2 = 0\)

The solution is of the form \(t_n = a_3\ n + b_3\ n + g_2^2\ n + d\)

Plugging in to initial conditions yields

\[ t_0 = 0 = a + g \]
\[ t_1 = 18 = 3a + 3b + 2g \]
\[ t_2 = 99 = 9a + 18b + 4g \]

Solving this system of equations yields \(a = 9, b = 3, g = -9\)

Therefore, \(t_n = 9\cdot 3^n + 3n\cdot 3^n - 9\cdot 2^n = (n + 3)\cdot 3^n + 1 - 9\cdot 2^n\)
3. Let $m$ be the missing integer. We let $S \subseteq \{0, \ldots, n\}$ be a set of candidates for $m$, and $A^* \subseteq \{1, \ldots, n\}$ be a set of subscripts of $A$ such that if $x \in S \backslash \{m\}$, then there is an $i \in A^*$ such that $A[i] = x$.

Initially, $S = \{0, \ldots, n\}$ and $A^* = \{1, \ldots, n\}$. As elements are removed from $S$, the corresponding subscripts are removed from $A^*$.

\[ S := \{0, \ldots, n\} \]
\[ A^* := \{1, \ldots, n\} \]

\textbf{for} $i := 1 \text{ to } \lceil \lg n \rceil \textbf{ do}\]
\textbf{let} $p$ \textbf{be the number of 1's in the} $i^{\text{th}} \text{ bit position of the elements of } S$
\textbf{sum} := 0

\textbf{for each } $i \in A^*$ \textbf{ do}

\textbf{if} the $j^{\text{th}} \text{ bit of } A[i] = 1 \textbf{ then } \textbf{sum} := \textbf{sum} + 1$

\textbf{if } $p = \textbf{sum} \textbf{ then remove from } S \text{ every element with a 1 in the}$
\textbf{if } $p = \textbf{sum} \textbf{ then remove from } S \text{ every element with a 1 in the}$
\textbf{else remove from } $S \text{ every element with a 0 in the}$
\textbf{else remove from } $S \text{ every element with a 0 in the}$
\textbf{th} bit, along with the corresponding indices
from $A^*$
from $A^*$

the element remaining in $S$ is $m$

The first pass examines $n$ bits of $A$.
The second pass examines $n/2$ bits of $A$.
The $i^{\text{th}}$ pass examines $n/(2^{i-1})$ bits of $A$.
The number of bits examined is

\[ \sum_{1 \leq i \leq \lceil \lg n \rceil} \frac{n}{(2^{i-1})} = n \sum_{1 \leq i \leq \lceil \lg n \rceil} \frac{1}{(2^{i-1})} \]

\[ = n \sum_{1 \leq i \leq \lceil \lg n \rceil} \frac{1}{(2^{i})} \]

\[ = n \sum_{0 \leq i \leq \lceil \lg n \rceil} \frac{1}{(2^{i})} \]

\[ = n \frac{1-(1/2)^{\lceil \lg n \rceil}}{1-(1/2)} \]

\[ = 2n \frac{1-(1/n)}{1-(1/2)} \]

\[ = 2n - 2 \]