1. (5 points) Imagine there are two teams, $A$ and $B$, who are playing until one team wins $n$ games, and the first team to win $n$ games is the winner (for the World series, $n=4$). There are no ties. Also assume there is a probability $p$, $0 \leq p \leq 1$, such that for each game, Pr{$A$ wins}=$p$, and the results of each game are independent.

Let $P(i,j)$, $0 \leq i, j \leq n$, be the probability that $A$ will win the series given that $A$ needs to win $i$ games to win the series and $B$ needs $j$ games to win the series. We can compute $P(n,n)$ according to:

$$P(0,j) = 1, \quad 1 \leq j \leq n$$

$$P(i,0) = 0, \quad 1 \leq i \leq n$$

$$P(i,j) = p \cdot P(i-1,j) + (1-p) \cdot P(i,j-1), \quad 1 \leq i, 1 \leq j$$

A) What is $P(4,4)$ for $p=0.42$? What is $P(7,7)$ for $p=0.42$?

B) In computing $P(i,j)$ according to the above recurrence, how many calls of $P$ are made? (Hint: When $P(i,j)$ makes its recursive calls, consider what happens to the pair $(i+j,i)$.) In computing $P(7,7)$ according to the above recurrence, how many calls of $P$ are made? Note that computing $P(2,2)$ requires 11 calls.

2. (3 points) Use summing factors to find a closed form solution for the recurrence:

$$M(n+1) = 3 \cdot M(n) + n, \quad n \geq 0$$

$$M(0) = 0$$

C.S.504
Solution for H.W. #6

1. A) With $p=0.42$, $P(4,4)=0.3294116066304$, and $P(7,7)=0.27695244363961946$.

B) Computing $P(i,j)$ requires
\[ 2 \cdot \binom{i+j}{i} - 1 \]
calls. Computing \( P(7,7) \) requires 6863 calls.

2. The recurrence \( M(n+1) = 3M(n) + n \) rewritten in the form
\[
M(n+1) = b_{n+1}M(n) + c_{n+1} \text{ yields } b_{n+1} = 3, c_{n+1} = n, \quad 0 \leq n.
\]
Using summing factors yields the solution to be
\[
M(n) = b_n \ldots b_1[ M(0) + \sum_{1 \leq i \leq n} c_i / (b_i \ldots b_1) ] = 3^n \sum_{1 \leq i \leq n} (i-1)/3^i
\]
Replacing \( i \) by \( i+1 \) in the \( \sum \) yields
\[
M(n) = 3^n \sum_{1 \leq i+1 \leq n} i/3^i + 1 = 3^n \sum_{0 \leq i \leq n-1} i/3^i
\]
Using results from class for solving \( \sum_{0 \leq i \leq n} i a^i \) yields
\[
M(n) = (3^{n+1}/4) \cdot [1/3 - n(1/3)^n + (n-1)(1/3)^n + 1]
= (3^n - 2n - 1)/4
\]