1. (5 points) Note that

\[
\begin{align*}
(9 \times 1) + 2 &= 11 \\
(9 \times 12) + 3 &= 111 \\
(9 \times 123) + 4 &= 1111 \\
(9 \times 1234) + 5 &= 11111
\end{align*}
\]

A) Express this property in terms of \(\Sigma\)-notation.

B) Generalize this property in terms of arbitrary integer radix \(b \geq 1\), not just base \(b=10\).

C) Prove the equation derived in B).

2. (2 points) For which value(s) of \(k\) is the binomial coefficient \(\binom{n}{k}\) a maximum, when \(n\) is a given positive integer? Justify your response.

3. (4 points) For random probing hashing (see Gonnet 3.3.3), show that

\[
E[A_n] = \frac{m}{n} (H_m - H_{m-n}) \leq \frac{1}{a} \ln \frac{1}{1-a} + \frac{1}{a}
\]

Hint: \(E[A_n]\) can be derived from \(E[A_n']\) by noting that in a successful search for \(key\), the number of probes necessary is the number of probes when \(key\) was inserted. When the \((i+1)^{st}\) is inserted, it was due to an unsuccessful search in a hash table of \(i\) keys. When \(key\) is successfully sought, it has a probability of \(1/n\) of having been the \(i^{th}\) key inserted, \(1 \leq i \leq n\).
C.S.504
Solution for H.W. #4

1. A) \[ 9\sum_{0 \leq k \leq n}(n-k)^*10^k + n + 1 = \sum_{0 \leq k \leq n}10^k \]
   B) \[(b-1)\sum_{0 \leq k \leq n}(n-k)^*b^k + n + 1 = \sum_{0 \leq k \leq n}b^k \]
   C) \[(b-1)^*\sum_{0 \leq k \leq n}(n-k)^*b^k + n + 1 \]
      \[= (b-1)^*\sum_{0 \leq k \leq n}b^k \]
      \[= (b-1)^*\sum_{0 \leq k \leq n}b^k - \frac{1}{(b-1)}[nb^n+2-(n+1)b^{n+1}+b]/(b-1)^2 + n + 1 \]
   \[= (b-1)^*n^*(b^{n+1-1})/(b-1) - [nb^n+2-(n+1)b^{n+1}+b]/(b-1) + n + 1 \]
   \[= [nb^n+2-nb^n+1-nb+n-nb^n+2-(n+1)b^{n+1}-nb-n+1]/(b-1) \]
   \[= (b^n+1-1)/(b-1) = \sum_{0 \leq k \leq n}b^k \]

2. \[ \binom{n}{k+1} = \frac{n-k}{k+1} \]
   so the binomial coefficients are increasing (the ratios between adjacent terms are ≥1) when \( k < \left\lfloor \frac{n}{2} \right\rfloor \), and the coefficients are decreasing (the ratios between adjacent terms are ≤1) when \( k < \left\lceil \frac{n}{2} \right\rceil \). The maximum occurs when \( k = \left\lfloor \frac{n}{2} \right\rfloor \) and \( k = \left\lceil \frac{n}{2} \right\rceil \).

3. \[ E[A_n] = \sum_{0 \leq i \leq n-1}(1/n)^*E[A_n] = (1/n)\sum_{0 \leq i < n-1}[1/(i+1/m)] \]
   \[= (m/n)\sum_{0 \leq i < n-1}1/(m-i) = (m/n)^*[1/m + ... + 1/(m-n+1)] \]
   \[= (m/n)^*(H_m - H_{m-n}) = (1/\alpha)^*(H_m - H_{m-n}) \]
   Using the bounds \( \ln_i \leq H_i \leq 1+\ln_i \), we get
   \[ E[A_n] = (1/\alpha)^*(H_m - H_{m-n}) \leq \alpha \cdot [1 + \ln m - \ln(m-n)] \]
   \[= (1/\alpha)^*\ln[m/(m-n)] + (1/\alpha) = (1/\alpha)^*\ln[1/(1-\alpha)] + (1/\alpha) \]