C.S.504  
H.W. #3  
Due: October 1, 1991

1. (2 points) In Section 3.1.1 of our text, what is \( E[A_n] \)
when \( \Pr\{A_n = i\} = \begin{cases} \frac{1}{2}^i & \text{if } 1 \leq i \leq n - 1 \\ \frac{1}{2}^{n-1} & \text{if } i = n \end{cases} \)

2. (3 points) Prove that
\[
\sum_{0 \leq k < n} (a_{k+1} - a_k) b_k = a_n b_n - a_0 b_0 - \sum_{0 \leq k < n} a_{k+1} (b_{k+1} - b_k)
\]
for \( n \geq 0 \).

3. (2 points) Suppose a program has declared

```plaintext
constant n = ...  
var r : array [1..n] of integer;  
i, m : 1..n;
```

Consider the following program segment to set \( m \) equal to the index of the maximum member of \( r \).

```plaintext
m := 1;  
for i := 2 to n do  
    if r[i] > r[m] then  
        m := i;
```

Suppose that \( r \) contains \( n \) distinct integers.

A) What is the worst-case number of executions of \( m := i \)? What problem instance would yield this number of executions?

Under the assumption that each permutation of the integers in \( r \) is equally likely (has probability \( 1/n! \)),

B) What is the expected number of executions of \( r[i] > r[m] \)?

C) What is the expected number of executions of \( m := i \)?

4. (4 points) We want to estimate the fraction of cylinders traversed by a "random" disk head seek or the number of places a "random" element of a "randomly shuffled" array must travel when sorting the array. More formally, we draw two integral points, \( here \) and \( there \), from a uniform
distribution over $[1..n]$, where uniform distribution means $\Pr\{here = i\} = 1/n$, for $1 \leq i \leq n$, and $\Pr\{there = i\} = 1/n$, for $1 \leq i \leq n$. We want to compute $E[|there-here|]$.

For a fixed value of $here$, we can define a probability model with the three events $\{there \in \text{left}, there = here, there \in \text{right}\}$, and we may assume:

- $\Pr\{there \in \text{left}\} = (here - 1)/n$
- $\Pr\{there = here\} = 1/n$
- $\Pr\{there \in \text{right}\} = (n - here)/n$

A quick check shows that these probabilities add up to 1. For each of these events, the values of $E[|there-here|]$ are:

- if $there \in \text{left}$ then $E[|there-here|] = (here)/2$
- if $there = here$ then $E[|there-here|] = 0$
- if $there \in \text{right}$ then $E[|there-here|] = (n - here + 1)/2$

A) For a fixed value of $here$, what is $E[|there-here|]$?

B) Removing the assumption of A), what is $E[|there-here|]$?
1. $2 - 2^{1-n}$

2. 
\[
\sum_{0 \leq k < n} (a_{k+1} - a_k) b_k = \sum_{0 \leq k < n} (a_{k+1} b_k - a_k b_k) \text{ distributive law}
\]
\[
= \sum_{0 \leq k < n} a_{k+1} b_k - \sum_{0 \leq k < n} a_k b_k \text{ associative law}
\]
\[
= \sum_{0 \leq k < n} a_{k+1} b_k - a_0 b_0 - \sum_{0 < k < n} a_k b_k
\]
\[
= \sum_{0 \leq k < n} a_{k+1} b_k - a_0 b_0 - \sum_{0 < k < n+1} a_{k+1} b_{k+1}
\]
\[
= a_n b_n - a_0 b_0 + \sum_{0 \leq k < n} a_{k+1} b_k - \sum_{0 \leq k < n} a_{k+1} b_{k+1}
\]
\[
= a_n b_n - a_0 b_0 - \sum_{0 \leq k < n} a_{k+1} (b_{k+1} - b_k)
\]

3. A) $n$. The worst-case instance is when $r$ is sorted in increasing order, that is, every $i$ is a relative maximum.

   B) $n$

   C) Σ(2≤i≤n)(1/i) = Hn - 1

4. A) Denoting $h$ by $h$, we note that for a fixed value of $h$,
\[
E[ | there-h | ] = [(h - 1)/n] *[h/2] + (1/n)*0 + [(n-h)/n]*[(n-h + 1)/2]
\]
\[
= (2h^2 + n^2 - 2h - 2nh + n)/(2n)
\]

   B) Each of the $n$ events, $here = i$, 1≤i≤n, occurs with probability 1/n.
\[
E[ | there-h | ] = \sum_{1 \leq h \leq n} (1/n)^* (2h^2 + n^2 - 2h - 2nh + n)/(2n)
\]
\[
= (1/2n^2)*\sum_{1 \leq h \leq n} (2h^2 + n^2 - 2h - 2nh + n)
\]
\[
= (1/2n^2)*[(4n^3 + 6n^2 + 2n)/6 + n^3 - n^2 - n - n^3 - n^2 + n^2]/6
\]
\[
= (1/2n^2)*(4n^3 + 6n^2 + 2n + 6n^3 - 6n^2 - 6n^3 - 6n^2 + 6n^3 - 6n^2 + 6n^3 - 6n^2)
\]
\[
\frac{(4n^3 - 4n)}{(12n^2)} = \frac{n}{3} - \frac{1}{3n}
\]