Due: September 17, 1991

The following three problems are from Cormen, Leiserson & Rivest, Introduction to Algorithms.

1. (2 points) Jack flips one fair coin (Pr{heads}=Pr{tails}=1/2), and Jill flips two fair coins. What is the probability that Jill obtains more heads than Jack?

2. (2 points) A deck of 10 cards, each bearing a distinct number from 1 to 10, is shuffled. Three cards are removed, one at a time from the deck. What is the probability that the cards are removed in sorted (increasing) order?

3. (4 points) You are given a possibly biased coin that, when flipped, produces a head with (unknown) probability $p$, where $0<p<1$. Show how a fair "coin flip" can be simulated by looking at multiple flips with the possibly biased coin. (Hint: Flip the coin twice and then output the result of the simulated flip or repeat the experiment.) Prove your answer.

4. (4 points) Problem 1.34 from Baase's Computer Algorithms: The first $n$ cells of the array $L$ contain integers sorted in increasing order. The remaining cells all contain some very large integer that we may think of as $\infty$ (e.g., maxint in Pascal). The array may be arbitrarily large, and you don't know $n$. Give an algorithm to find the position of a given integer $x$ ($x<$maxint) in the array in $O(lgn)$ time.

5. (3 points) Purdom & Brown's The Analysis of Algorithms gives the following exercise:

Show that

$$\frac{\ln\left(1 + \frac{t \left(-\ln(1-p)\right)}{\ln2}\right)}{-\ln(1-p)} = \frac{\ln2}{\ln2 + t \left(-\ln(1-p)\right)} \cdot \ln2$$

Either prove or give a counterexample to this exercise.
1.) Jill gets more heads if either of the following three events occurs:
- Jill gets 2 heads & Jack gets 0 heads
- Jill gets 1 head and Jack gets 0 heads
- Jill gets 2 heads and Jack gets 1 head
The three mutually exclusive events have probabilities \((1/2)^3\), \((1/2)^2\) and \((1/2)^3\). The probability of Jill getting more heads is the sum of these probabilities, or \(1/2\).

2) The probability of any sequence of three distinct digits being drawn is \((1/10)*(1/9)*(1/8)=1/720\), so all such sequences are equiprobable. Therefore, the experiment is equivalent to asking the probability that a random permutation (drawn from a uniform distribution) of \{1,2,3\} will be increasing. Each of the six permutations being equiprobable, the answer is \(1/6\).

3) The Hint suggests that we seek an experiment with three possible outcomes, \{output_head, output_tail, repeat_the_experiment\}, and \(Pr\{output_head\}=Pr\{output_tail\}=1/2\). We want to find equiprobable results from flipping a possibly biased coin twice. If the two flips yield one head and one tail, then the outcomes (head,tail) and (tail,head) are equiprobable, each with a probability \(p*(1-p)\).

\[\begin{align*}
\text{repeat forever} \\
& f_1 \leftarrow \text{result of a flip} \\
& f_2 \leftarrow \text{result of a flip} \\
& \text{if } (f_1=\text{head} \text{ and } f_2=\text{tail}) \text{ then return("output_head")} \\
& \text{if } (f_1=\text{tail} \text{ and } f_2=\text{head}) \text{ then return("output_tail")}
\end{align*}\]

4) \(i := 1;\)
\[\begin{align*}
\text{while } L[i] < x \text{ do } i := 2*i; \quad \text{§ do binary search for } x \text{ in } L[(i \text{ div } 2)+1..i] \\
\end{align*}\]
Since \(i\) starts at 1 and can be doubled at most \(\log_2 n\) times until reaching \(n\), and \(L[n+1] > x\), step § can be executed at most \(\lceil \log_2 n \rceil\) times. The binary search of step • requires \(O(\log_2 n)\) steps.
5. The equality is incorrect, as the instance with $p = 0.5$ and $t = 1$ shows:

```scheme
(define (lhs p t)
    (expt 2 (/ (log (1+ (/ (* t (- (log (- 1 p))))
                    (log 2))))
           (- (log (- 1 p))))))

>>> (lhs 0.5 1)
2.0

(define (rhs p t)
    (expt (/ (* t (- (log (- 1 p))))
            (+ (log 2) (* t (- (log (- 1 p))))))
           / (log 2)
           (- (log (- 1 p)))))

>>> (rhs 0.5 1)
0.5
```