1. (25 points) Describe the simplest closed form solution for \( f(n) \), defined by the recurrence

\[
f(n) = 2f\left(\frac{n}{2}\right) + n \log n.
\]

You don’t have to solve for the coefficients of the closed form solution; you only need to describe \( f(n) \) as a closed form function of \( n \) in terms of some undetermined coefficients which only depend upon initial conditions.
2. (25 points) Find a closed form for \( \sum_{k \geq 0} \frac{k-1}{2^k} \).
3. (25 points) Consider the following algorithm to construct a certain kind of tree in an array $A[1..n]$. The initial invocation is $\text{BUILD}(1,n)$.

```c
void BUILD(int lo, hi)
{
    int mid;
    if (hi > lo) {
        mid = (lo + hi) / 2;
        BUILD(lo, mid);
        BUILD(mid + 1, hi);
        if ($A[mid] > A[hi]$) SWAP(lo, mid, hi);
    }
}
```

We are interested in determining $f(n)$, the worst-case number of comparisons of elements of $A$ (assume that SWAP does not compare elements of $A$).

(A) Assuming that $n$ is a power of 2, derive a recurrence for $f(n)$.

(A) Assuming that $n$ is a power of 2, find a closed form for $f(n)$.
4. (25 points) The analysis of a certain sorting algorithm involves the following sum:

$$
\sum_{k=0}^{\lfloor \log_2 n \rfloor - 1} 2^{k+1} \log_2 \frac{n}{2^k}
$$

Assume that $n$ is a power of 2, and find a closed form for the sum.
1. Substituting $2^k$ for $n$ yields $f(2^k) = 2f\left(\frac{2^k}{2}\right) + 2^k \lg 2^k = 2f(2^{k-1}) + k2^k$. Substituting $a_k$ for $f(2^k)$ yields the linear first-order nonhomogeneous recurrence $a_k - 2a_{k-1} = k2^k$, with the characteristic equation $(x - 2)^3 = 0$ with characteristic root 2 of multiplicity 3. Hence, the solution (for $a_k$) is of the form $a_k = f(2^k) = a\cdot 2^k + b\cdot k2^k + c\cdot 2^{2k}$. Since $k = \lg n$, $f(n) = a\cdot n + b\cdot \lg n + c\cdot (\lg n)^2$.

2. 

$$
\sum_{k=0}^{n-1} k \cdot 2^k = \sum_{k=0}^{n-1} k \cdot \left(\frac{1}{2}\right)^k - \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k = \lim_{n \to \infty} \frac{1}{2} - \frac{(n+1)(\frac{1}{2})^{n+1} + n\cdot \frac{n+2}{2}}{\left(-\frac{1}{2}\right)^2} - \lim_{n \to \infty} \frac{1}{2} - \frac{1}{2} = 2 - 2 = 0
$$

3. (A) \( f(n) = \begin{cases} 0, & \text{if } n \leq 1 \\ 2f(n/2) + 1, & \text{if } n > 1 \end{cases} \)

(B) Substituting $2^k$ for $n$, $f(2^k) = 2f\left(\frac{2^k}{2}\right) + 1 = 2f(2^{k-1}) + 1$, and substituting $x_k$ for $f(2^k)$, yields the first-order linear nonhomogeneous recurrence $x_k = 2x_{k-1} + 1$ which admits the characteristic equation $(x - 2)(x - 1) = 0$ with characteristic roots 1 and 2, each of multiplicity 1. The solution is $f(2^k) = x_k = a\cdot 2^k + b$. Substituting $k = \lg n$,

$$
f(2^k) = f(n) = a\cdot n + b, \quad \text{and plugging in initial conditions, } f(1) = 0 = a + b \Rightarrow a = -b \quad \text{and} \quad f(2) = 2f(1) + 1 = 2a + b = -2b \Rightarrow b = -1, \quad \text{so the final solution is } f(n) = n - 1.
$$

4. 

$$
\sum_{k=0}^{n-1} 2^{k+1} \cdot \lg n \cdot 2^k = \sum_{k=0}^{n-1} 2^{k+1} \cdot \lg n - \sum_{k=0}^{n-1} k2^{k+1} = 2\lg n \sum_{k=0}^{n-1} 2^k - 2 \sum_{k=0}^{n-1} k2^k
$$

$$
= 2\lg n \left(2^n - 1\right) - 2 \left(2 - \lg n 2^n + (\lg n - 1)2^{n+1}\right)
$$

$$
= 2n \lg n - 21\lg n - 4 + 2n \lg n - 4n(\lg n - 1) = 4n - 21\lg n - 4
$$