Final Exam

Date: April 22, 1998
All documentation permitted

2. (20 points) Find a closed form for \( \sum_{k=1}^{n} \binom{n}{k} 2^{k-n} \) for \( n \geq 0 \).
3. (30 points) In a binary tree (not necessarily full), every node has 0, 1 or 2 children. Nodes with 1 or 2 children are internal nodes. The binary trees with 0, 1 and 2 internal nodes are

Give a closed form for the generating function \( B(z) = 1 + 2z + 6z^2 + \ldots \) such that \( [z^n]B(z) \) is the number of binary trees with \( n \) internal nodes. You don’t need to find a closed form for the coefficients of \( B(z) \).
4. (25 points) Suppose you flip a fair coin until the first heads, and let random variable
X denote the number of flips until the first occurrence of heads. For example, if you flip
Tail, Tail, Head, then X=3. For positive integers p≥q>0, give a closed form for
\[ f(p,q) = \Pr\{ p \geq X \geq q \} \]. For example, f(1,1)=Pr\{X=1\}=1/2.
2. \[ \sum_{k=1}^{n} \binom{n}{k} 2^{k-n} = 2^{-n} \left( \sum_{k=0}^{n} \binom{n}{k} 2^{k} - \binom{n}{0} 2^{0} \right) = 2^{-n} \left( \sum_{k=0}^{n} \binom{n}{k} 2^{k} \right) \frac{1}{2} = 2^{-n} \left( 3^{n} - 1 \right) \]

3. A binary tree has 0 internal nodes (counted by 1), or a root with one binary subtree (counted by \( B(z) \)), or a root with two binary subtrees (each counted by \( B(z) \)). Thus,
\[
B(z) = 1 + zB(z) + zB(z)B(z) \]

\[
0 = zB(z)B(z) + (z-1)B(z) + 1 \]

\[
B(z) = \frac{1 - z \pm \sqrt{1 - 6z}}{2z} \]

To choose the correct root, we note that \( B(0) = 1 \). Substituting \( z = 0 \) in the generating function yields \( B(z) = \frac{1 + \sqrt{1}}{0} \), which shows that we must choose the negative root.

\[
B(z) = \frac{1 - z - \sqrt{1 - 6z}}{2z} \]

4. \[
\hat{f}(p,q) = \sum_{p \geq k \geq q} (1/2)^{k-1}(1/2) = \sum_{p \geq k \geq 0} (1/2)^{k} - \sum_{q \geq 0} (1/2)^{k} \]

\[
\frac{1 - (1/2)^{p+1}}{1 - (1/2)} - \frac{1 - (1/2)^{q}}{1 - (1/2)} = \left( \frac{1}{2} \right)^{q-1} - \left( \frac{1}{2} \right)^{p} \]