Final Exam

Date: December 11, 1996
All documentation permitted

1. (25 points) Solve the recurrence

\[(n + 1)t_n = nt_{n-1} + n + 1\]

with the initial condition \[t_0 = 1.\]
2. (25 points) Find a closed form for the generating function \( G(z) = \sum_{n} g_n z^n \) whose coefficients are related by

\[
g_n = \begin{cases} 
0, & \text{if } n = 0 \\
1, & \text{if } n = 1 \\
3g_{n-1} + \sum_{k} g_k g_{n-k}, & \text{if } n > 1 
\end{cases}
\]

You don’t have to extract the coefficients \( g_n \).
3. (25 points) For positive integer $n$ written in decimal notation, let $\nu(n)$ denote the length of the maximal string of even digits of $n$ starting from the right (the least significant digit). For example, $\nu(53481) = 0$, $\nu(53482) = 3$, and $\nu(42) = 2$. Drawing $n$ from a uniform distribution over a contiguous sequence of very large numbers (say all 100,000 digit numbers), what is the expected value of $\nu(n)$?
4. (25 points) Leonhard Euler showed that the infinite product
\[
\prod_{k \geq 0} \left(1 + z^{2^k}\right) = (1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \ldots \text{ is equal to } \frac{1}{1 - z}.
\]

A) Give a closed form for \( \prod_{t \geq 0} \left(1 + z^{2^t}\right) \), where \( t \) is any non-negative integer.

B) Interpret (explain in words) both sides of the identity 
\[
\frac{1}{1 - z} = (1 + z)(1 + z^2)(1 + z^4)(1 + z^8) \ldots \text{ in terms of the binary representation of non-negative integers.}
\]
1. \((n+1)t_n = nt_{n-1} + n + 1\) is a first-order linear nonhomogeneous recurrence. Using summation factors with \(b_n = \frac{n}{n+1}\) and \(c_n = 1\), we use \(b_n \ldots b_1 = \frac{1}{n+1}\) to solve for
\[
t_n = b_n \ldots b_1 \left( t_0 + \sum_{1 \leq k \leq n} \frac{c_k}{b_k \ldots b_1} \right) = \frac{1}{n+1} \left( 1 + \sum_{n \geq k \geq 1} \frac{1}{k(k+1)} \right) = \frac{1}{n+1} \left( n + 1 + \sum k \right) \]
\[
= 1 + \frac{1}{n+1} \frac{(n+1)n}{2} = 1 + \frac{n}{2}.
\]
2. The recurrence can be written
\[
g_n = 3g_{n-1} + \sum_k g_k g_{n-k} + [n = 1].
\]
Multiplying by \(z^n\) and summing over all \(n\),
\[
G(z) = \sum_n g_n z^n = \sum_n 3g_{n-1} z^n + \sum_n \sum_k g_k g_{n-k} z^n + \sum_n [n = 1] z^n
\]
\[
G(z) = 3zG(z) + \left( G(z) \right)^2 + z = \frac{1 - 3z \pm \sqrt{9z^2 - 10z + 1}}{2}
\]
Since \(G(0) = g_0 = 0\), we know that the negative root is correct, so
\[
G(z) = \frac{1 - 3z - \sqrt{9z^2 - 10z + 1}}{2}.
\]
3. Reading \(n\) from the right, we define \textit{success} to be the reading of an odd digit. For any digit of \(n\), \(\Pr\{\text{success}\} = \frac{1}{2}\). Since the digits are independent, the waiting time until \textit{success} is geometrically distributed with expected value \(\frac{1}{p} = \frac{1}{\frac{1}{2}} = 2\). Note the slight error in the answer since there is a probability of greater than 0 of the waiting time exceeding 100,000.

4. A) \(\prod_{i \geq 0} (1 + z^{2^i}) = \sum_{z^k} k = \frac{z^{2^k+1} - 1}{z - 1}, \) so \n is represented exactly once on the left side of the equality. The term \((1 + z^{2^i})\) \(k \geq 0\) states the \(k\) \(th\) bit is or is not set in the binary representation of a number, and this decision is made independently of all the other bits. Each arrangement of bits appears exactly once.