Consider the following program to find the two largest keys in array \( L[1..n] \) (assume \( n \geq 2 \)):

\[
\begin{align*}
\text{if } L[1] > L[2] & \text{ then } \max := L[1] \\
& \quad \text{second} := L[2] \\
\text{else } \max := L[2] \\
& \quad \text{second} := L[1]
\end{align*}
\]

\[
\text{for } i := 3 \text{ to } n \text{ do if } L[i] > \text{second} \text{ then}
\]

\[
\begin{align*}
& \quad \text{if } L[i] > \max \text{ then } \text{second} := \\
& \quad \max := L[i] \\
& \quad \text{else } \text{second} := L[i]
\end{align*}
\]

A) How many key comparisons does this algorithm do in the worst case? What is a worst case input?

B) How many key comparisons does this algorithm do in the average case, assuming the keys are distinct and every input permutation of the keys is equally likely?
2. (35 points) Compute $E[A_n]$ for optimal sequential search (Gonnet 3.1.4, pg. 34) for the distribution $p_j = z^*(n - i + 1), 1 \leq i \leq n$, for an appropriate choice of $z$. 
3. (35 points) Let \((a_1, a_2, \ldots, a_n)\) be a permutation of \((1, 2, \ldots, n)\). The total net distance traveled in sorting is
\[ |a_1 - 1| + |a_2 - 2| + \ldots + |a_n - n| \]
For permutation \((a_1, a_2, a_3, a_4) = (3, 1, 4, 2)\), these values are
\(|3 - 1| = 2, |1 - 2| = 1, |4 - 3| = 1, \text{ and } |2 - 4| = 2\).
Assume a permutation is drawn from a uniform distribution over the set of all permutations. That is, the probability of drawing any permutation is \(1/n!\). It follows that \(\Pr\{a_i = j\} = 1/n\) for \(1 \leq i, j \leq n\), that is, each element has a probability \(1/n\) of being in any position.

A) Write an expression for \(E[|a_j - j|], 1 \leq j \leq n\), as a function of the terms \(|1 - j|, |2 - j|, \ldots, |n - j|\).

B) Express the sum \(|1 - j| + |2 - j| + \ldots + |(j - 1) - j|\) in terms of binomial coefficients. (Hint: In the event that element \(j\) is permuted to position \(i\), \(1 \leq i \leq j\), it must move through position \(k\), \(i < k \leq j\). How many such pairs \((i,k)\) are possible?)

C) Express the sum \(|(j + 1) - j| + |(j + 2) - j| + \ldots + |(n) - j|\) in terms of binomial coefficients.

D) Use your expression from part A) with your results for parts B) and C) and binomial identities discussed in class to find the expected total net distance traveled, that is, the expected value of \(|a_1 - 1| + |a_2 - 2| + \ldots + |a_n - n|\).
CS504
Solutions to Midterm Exam

1. A) When the keys are sorted in increasing order, the algorithm does
2n -3 comparisons, 1 for the initial "if L [1]>L [2] then...", and 2(n - 2) for the for-loop.
B) The program always uses 1 comparison for the "if L [1]>L [2] then..." statement, and then the for-loop does 1 or 2 comparisons, depending upon the condition "L [i]>second". Assuming all permutations are equally likely, Pr{L [i]>second}=2/i, so the expected number of comparisons for a fixed i is 2*(2/i) + 1*(i - 2)/i. As a check, note that the probabilities of the two events, (2/i) and (i-2)/i, add up to 1. Our answer is
1+∑(3≤i≤n)[ 2*(2/i) + 1*(i-2)/i ]
= 1 + 4* ∑(3≤i≤n)(1/i) + ∑(3≤i≤n)1 - 2*∑(3≤i≤n)(1/i)
= 1 + (n-2) + 2*[∑(1≤i≤n)(1/i) - 3/2] = n - 4 + 2Hn + n - 4 + 2ln n

2. Solve for z by 1=∑(1≤i≤n)p_i=∑(1≤i≤n) z*(n-i+1)
= z[ ∑(1≤i≤n)n - ∑(1≤i≤n)i + ∑(1≤i≤n)1] = z[n^2 - n(n+1)/2 + n] = (n+1)n/2
z = 2/[n (n+1)]
E[A_n] = ∑(1≤i≤n)j*i*Pr{A_n=i} = ∑(1≤i≤n)j*i*2*(n-i+1)/(n*(n+1)]
= 2/[n*(n+1)]*∑(1≤i≤n)j*(n-i+1) = ... = (n+2)/3

3. A) E[ |aj-j | ] = ∑(1≤i≤n) |i-j | * Pr{aj=i}
= (1/n) * ∑(1≤i≤n) |i-j |
= (1/n) * [ |1-j | + |2-j | + ... + |n-j | ]
B) C(j 2)
C) C(n-j+1 2)
D) E[ |aj-j | ] = (1/n) * [ |1-j | + |2-j | + ... + |n-j | ]
= (1/n) * [ C(j 2) + C(n-j+1 2) ]
The expected total net distance traveled is
Σ(1≤j≤n) (1/n) * [ C(j 2) + C(n-j+1 2) ]
= (1/n) * [ Σ(1≤j≤n) C(j 2) + Σ(1≤j≤n) C(n-j+1 2) ]
= (1/n) * [ Σ(1≤j≤n) C(j 2) + Σ(1≤j≤n) C(j 2) ]
\[
\begin{align*}
&= \frac{1}{n} \left[ \sum_{0 \leq j \leq n} \binom{j}{2} + \sum_{0 \leq j \leq n} \binom{n-j+1}{2} \right] \\
&= \frac{1}{n} \left[ \binom{n+1}{3} + \binom{n+1}{3} \right] \\
&= \frac{[2 \cdot (n+1) \cdot n \cdot (n-1)]}{[3 \cdot 2 \cdot n]} \\
&= \frac{(n^2 - 1)}{3}
\end{align*}
\]

Note that the average element travels a distance \( n/3 \).