Due: April 22/23, 1998

1. (12 points) Consider the following program fragment operating on array $A[n]$:

   ```
   max = -\infty;
   for (i=0; i<n; i++)
     if (A[i] > max) max = A[i];
   ```

Assume that $A$ contains a permutation of $\{1, \ldots, n\}$ drawn from a uniform distribution over the set of all such permutations. Give a closed form for each of the following answers.

   (A) What is the expected number of times $\max = A[i]$ is executed?
   (B) What is the variance of the number of times $\max = A[i]$ is executed? You may use the definition of equation (6.61) on page 277 of our text.
   (C) What is the standard deviation of the number of times $\max = A[i]$ is executed?
   (D) What is the probability that $\max = A[i]$ is executed $n$ times?
   (E) For $n=10$, what is the probability that $\max = A[i]$ is executed 10 times?
   (F) For $n=10$, use Chebyshev’s inequality to bound the probability that $\max = A[i]$ is executed $n$ times?

2. (3 points) We know that if $f$ is a function from real numbers to real numbers and $X$ is a random variable, then $f(X)$ is a random variable. Prove or give a counterexample to the following.

   **Conjecture:** For any functions $f$ and $g$ and any independent random variables $X$ and $Y$, random variables $f(X)$ and $g(Y)$ must be independent.

3. (8 points) Knuth describes the following rather silly algorithm to sort a list $A$ of $n$ distinct numbers:

   ```
   procedure pokeysort(n : integer)
     begin if $n > 1$ then repeat
       $k \leftarrow$ random element of $\{1, \ldots, n\}$
       swap $A[k]$ $\leftrightarrow$ $A[n]$
       pokeysort(n-1)
     until $A[n-1]$ $\leq$ $A[n]$
   end
   ```

For $n \geq 2$, define random variable $X_i$ to be the number of recursive invocations of $pokeysort$, not counting the initial invocation. $X_1 = 0$. For $n=2$ and $A[1]=5$ and $A[2]=8$, if each invocation of $pokeysort(2)$ sets $k$ to 1, then $X_2 = 2$. For the same input, if the invocation of $pokeysort(2)$ sets $k$ to 2, then $X_2 = 1$.

   (A) What is $E[X_2]$?
   (B) Define a recurrence for $E[X_n]$ in terms of $E[X_{n-1}]$ for $n \geq 3$.
   (C) What is $E[X_n]$?