Homework 3

Due: Beginning of class, October 2, 2000

Part I.

1. Show the following two binomial sums, which can all be derived by tweaking the Vandermonde convolution identity:

   (a) \[ \sum_i \binom{p}{m+i} \binom{d}{n+i} = \binom{p+d}{p-m+n}, \quad p \in \mathbb{Z}_0, \ m, n \in \mathbb{Z}. \]

   (b) \[ \sum_i \binom{p-i}{m} \binom{d+i}{n} = \binom{p+d+1}{m+n+1}, \quad n \geq d, m, p \in \mathbb{Z}_0. \]

   (c) Compute: \[ \sum_k \left( \binom{n}{k} \right)^2. \]

2. Let an experiment consist of flipping a fair coin 5 times. Define the random variable \( C \) as the number of times the sequence changes. We count a change when Tails is followed by Heads, and vice versa. For example, the following sequences have the indicated value of \( C \): HHHHH – 0; TTHTT – 2; HTHTH – 4. What is the PMF of this random variable?

3. In the handout of lecture 3 I list four basic properties of the expectation operation. Prove them from the definition of expectation.

4. In a card game where five-card hands are dealt from a well-shuffled deck of 52 cards, compute:

   (a) What is the probability that a five-card hand has at least one card of each suit?

   (b) What is the probability that a five-card hand has exactly 3 kings?

   (c) Explain why the answers in parts (a) and (b) do not depend on the number of hands dealt? Rephrase part (a) so it will depend on the number, and solve your problem. Note that there are several ways to change the problem as specified.

5. Peter wants a new car, and his dad, who is a tennis fiend, wants Peter to improve his tennis play. He told Peter that if Peter wins two games in a row in a sequence of three games – his opponents alternating between his younger brother Jack and sister Anna, he gets a new car. Should Peter choose to play Jack or Anna first? It is known that Peter’s probabilities of beating Anna and Jack are 0.1 and 0.8 respectively.