CS CS2223 - D Term 2009: Algorithms Prof. Carolina Ruiz

Homework 2

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Problem 1: (32 Points) Suppose you have algorithms with the running times listed below. Assume these are the exact number of operations performed as a function of the input size n. Suppose you have a computer that can perform 10^{10} operations per second, and you need to compute results in at most

- (8 Points) 1 hour
- (8 Points) 1 week
- (8 Points) 1 year (you can assume that 1 year = 365 days)
- (8 Points) 1 century

For each of the algorithms, what is the largest input size n for which you would be able to get the result within the given time limit? That is, your solutions to this exercise should contain a table like the one below together with explanations for each of the values you provide in your table.

Solution 1: The following are the number of operations per given time:

$$10^{10} ops/second \times 60 * 60 \; seconds = 3.60 \times 10^{13} ops \; in \; 1 \; hour$$
$$10^{10} ops/second \times 60 * 60 * 24 * 7 \; seconds = 6.05 \times 10^{15} ops \; in \; 1 \; week$$
$$10^{10} ops/second \times 60 * 60 * 24 * 365 \; seconds = 3.15 \times 10^{17} ops \; in \; 1 \; year$$
$$10^{10} ops/second \times 60 * 60 * 24 * 365 * 100 \; seconds = 3.15 \times 10^{19} ops \; in \; 1 \; century$$

The numbers in the table below were calculated as follows. Let t be the number of operations that can be performed within the time limit.

Table 1: Largest input size n versus running time

Complexity	1 hour	1 week	1 year	1 century
$518n$	6.94×10^{10}	1.16×10^{13}	6.08×10^{14}	6.08×10^{16}
n^5	5.14×10^{2}	1.43×10^{3}	3.16×10^{3}	7.93×10^{3}
$log_{10}n$	$1\times10^{3.6\times10^{13}}$	$1\times10^{6.05\times10^{15}}$	$1\times10^{3.15\times10^{17}}$	$1\times10^{3.15\times10^{19}}$
4^n	22	26	29	32
$\sqrt[3]{n}$	4.66×10^{40}	2.21×10^{47}	$3.13.\times10^{52}$	3.13×10^{58}

1. If $518n = t$ then $n = t/518$.

   For a time limit of 1 hour, t = 3.60×10^{13} ops. Hence $n = 3.60 \times 10^{13}/518 = 6.94 \times 10^{10}$. Similarly for the other time limits.

2. If $n^5 = t$ then $n = \sqrt[5]{t}$.

   For a time limit of 1 hour, t = 3.60×10^{13} ops. Hence $n = \sqrt[5]{3.60 \times 10^{13}} = 5.14 \times 10^2$. Similarly for the other time limits.

3. If $log_{10}n = t$ then $n = 1 \times 10^t$.

   For a time limit of 1 hour, t = 3.60×10^{13} ops. Hence $n = 1 \times 10^{3.6\times10^{13}}$. Similarly for the other time limits.

4. If $4^n = t$ then $n = \log_4 t$.

   For a time limit of 1 hour, t = 3.60×10^{13} ops. Hence $n = \log_4(3.60 \times 10^{13}) = 22$. Similarly for the other time limits.

5. If $\sqrt[3]{n} = t$ then $n = t^3$.

   For a time limit of 1 hour, t = 3.60×10^{13} ops. Hence $n = (3.60 \times 10^{13})^3 = 4.66 \times 10^{40}$. Similarly for the other time limits.

We can now reorganize the rows in the previous table to make explicit the increase in time complexity and hence the reduction in the size of the input allowed in the time limit:

Table 2: Largest input size n versus running time, sorted by time complexity

Complexity	1 hour	1 week	1 year	1 century
$log_{10}n$	$1\times10^{3.6\times10^{13}}$	$1\times10^{6.05\times10^{15}}$	$1\times10^{3.15\times10^{17}}$	$1\times10^{3.15\times10^{19}}$
$\sqrt[3]{n}$	4.66×10^{40}	2.21×10^{47}	$3.13.\times10^{52}$	3.13×10^{58}
$1000n$	3.30×10^{3}	1.82×10^{4}	6.80×10^{4}	3.16×10^{5}
n^4	2.45×10^{3}	8.82×10^{3}	2.37×10^{4}	7.49×10^{4}
3^n	28	33	36	40

Problem 2: (40 Points) Problem 3 In each of the following cases, determine whether $f(n) = O(g(n))$, or $f(n) = \Omega(g(n))$, or both (i.e., $f(n) = \Theta(g(n))$), or none of the above. Prove your answers decisively.

Solution 2:

1

$$f(n) = n^2(\log n),\ g(n) = n^2$$
$$\lim_{n\to+\infty} \frac{f(n)}{g(n)} = \lim_{n\to+\infty} \frac{n^2(\log n)}{n^2} = \lim_{n\to+\infty} \log n = +\infty$$
Hence, $f(n) = \Omega(g(n))$ and $f(n) \neq O(g(n))$

2

$$f(n) = n^2(\log n),\ g(n) = n^3$$
$$\lim_{n\to+\infty} \frac{f(n)}{g(n)} = \lim_{n\to+\infty} \frac{n^2(\log n)}{n^3} = \lim_{n\to+\infty} \frac{\log n}{n} = \lim_{n\to+\infty} 1/n \text{ (by applying de l'Hôpital's rule) } = 0$$
Hence, $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$

3

$$f(n) = 3^n,\ g(n) = 3^{n+3}$$
$$\lim_{n\to+\infty} \frac{f(n)}{g(n)} = \lim_{n\to+\infty} \frac{3^n}{3^{n+3}} = \lim_{n\to+\infty} \frac{3^n}{3^n * 3^3} = \frac{1}{3^3}$$
Hence, $f(n) = \Theta(g(n))$

Just to practice working with the definitions of O, Ω, and Θ directly, we include below an alternate proof of the fact that $f(n) = \Theta(g(n))$:

We need to prove that there exist constants $c > 0, k > 0, n_0 \geq 0$ such that $k * g(n) \leq f(n) \leq c * g(n)$ for all $n \geq n_0$. Since $3^{n+3} = 3^3 * 3^n$, then if we take $k = 1/(3^3)$, $c = 1$, and $n_0 = 1$ we have that: $k * 3^{n+3} \leq 3^n \leq c * 3^{n+3}$ since $(1/(3^3)) * 3^3 * 3^n \leq 3^n \leq 1 * 3^3 * 3^n$ because $3^n \leq 3^n, \leq 3^3 * 3^n$ for all $n \geq 1$.

4

$$f(n) = 3^{3n},\ g(n) = 3^n$$

$$\lim_{n\to+\infty} \frac{f(n)}{g(n)} = \lim_{n\to+\infty} \frac{3^{3n}}{3^n} = \lim_{n\to+\infty} \frac{3^n * 3^n * 3^n}{3^n} = \lim_{n\to+\infty} 3^n * 3^n = +\infty$$

Hence, $f(n) = \Omega(g(n))$ and $f(n) \neq O(g(n))$

5

$$f(n) = \sqrt{n},\ g(n) = \log n$$

$$\lim_{n\to+\infty} \frac{f(n)}{g(n)} = \lim_{n\to+\infty} \frac{\sqrt{n}}{\log n} = \lim_{n\to+\infty} \frac{(1/2) * n^{-1/2}}{1/n} \text{ (by applying de l'Hôpital's rule)}$$

$$= \lim_{n\to+\infty} \frac{n * n^{-1/2}}{2} = \lim_{n\to+\infty} \frac{\sqrt{n}}{2} = +\infty$$

Hence, $f(n) = \Omega(g(n))$ and $f(n) \neq O(g(n))$

One can generalize some of the facts proven above as follows:

- (Generalization from parts 1 and 2 above:) One can prove that for any constant k, $n^k = O(n^k(\log n))$ and $n^k(\log n) = O(n^{k+1})$, but $n^k \neq \Omega(n^k(\log n))$ and $n^k(\log n) \neq \Omega(n^{k+1})$.
- (Generalization from part 3 above:) One can prove that for any constants a and $b \geq 0$, $a^n = \Theta(a^{n+b})$.
- (Generalization from part 3 above:) One can prove that for any constants a and $b > 1$, $a^n = O(a^{b*n})$, but $a^n \neq \Omega(a^{b*n})$

Problem 3: (28 Points) Construct a formal mathematical proof of each of the following two properties using the definitions of O, Ω, and Θ.

- **(14 points) Transpose symmetry of O and Ω:** $f(n)$ is $O(g(n))$ if and only if $g(n)$ is $\Omega(f(n))$.
- **(14 points) Symmetry of Θ:** $f(n)$ is $\Theta(g(n))$ if and only if $g(n)$ is $\Theta(f(n))$.

Solution 3:

- **Transpose symmetry of O and Ω:**
  - See a proof of this fact in the Solutions to Quiz1 CS2223 B05 at: http://web.cs.wpi.edu/∼cs2223/b05/Exams/Quiz1/
  - Also, Solved Exercise 2 on p. 66 of the textbook presents a proof of this fact.
- **Symmetry of Θ:**

  See a proof of this fact in the Solutions to Exam1 CS2223 B05 at:

  http://web.cs.wpi.edu/∼cs2223/b05/Exams/Exam1/