Practice Examination #2 Solutions

Note: This practice examination contains more problems than would typically be found on a real examination.

PROBLEM 1
Consider the following theorem:

\[(p \rightarrow q) \rightarrow r \Rightarrow p \rightarrow (q \rightarrow r)\]

Part A
Which of the following proof techniques could be used?

1. Truth Table? Yes.
2. Direct Proof? Yes.
3. Indirect Proof? Yes.

Part B
Prove the theorem using any one of the methods above. Give a reason for each step.

Direct Proof:

<table>
<thead>
<tr>
<th>#</th>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( p \rightarrow q ) \rightarrow r</td>
<td>Assumption</td>
</tr>
<tr>
<td>2</td>
<td>( \neg(p \rightarrow q) \lor r )</td>
<td>Implication</td>
</tr>
<tr>
<td>3</td>
<td>( \neg(\neg p \lor q) \lor r )</td>
<td>Implication</td>
</tr>
<tr>
<td>4</td>
<td>( (\neg p \land q) \lor r )</td>
<td>DeMorgan</td>
</tr>
<tr>
<td>5</td>
<td>( (p \land \neg q) \lor r )</td>
<td>Double negation</td>
</tr>
<tr>
<td>6</td>
<td>( (p \lor r) \land (\neg q \lor r) )</td>
<td>Distributive</td>
</tr>
<tr>
<td>7</td>
<td>( \neg q \lor r )</td>
<td>Simplification</td>
</tr>
<tr>
<td>8</td>
<td>( \neg p \lor (\neg q \lor r) )</td>
<td>Addition</td>
</tr>
<tr>
<td>9</td>
<td>( p \rightarrow (\neg q \lor r) )</td>
<td>Implication</td>
</tr>
<tr>
<td>10</td>
<td>( p \rightarrow (q \rightarrow r) )</td>
<td>Implication</td>
</tr>
</tbody>
</table>

Thus, the theorem is proven.

PROBLEM 2
Consider the following statement: For any positive integer \( k \), the sum of the digits in the decimal expansion of \( k \mod 9 \) equals \( k \mod 9 \). We will prove it by induction on the number of digits in the decimal expansion of \( k \).

Part A
Let \( n \) be the number of digits in the decimal expansion of \( k \), \( n = \text{DIGIT}(k) \). Complete the proposition \( P(n) \) which must be proven for all \( n \).
\( P(n) \): “If \( k = \sum_{i=0}^{n-1} a_i 10^i \), then \( k \equiv \sum_{i=0}^{n-1} a_i \pmod{9} \).”

**Part B**
Prove the basis.
\( P(1) \): “If \( k = a_0 10^0 \), then \( k \equiv a_0 \equiv k \pmod{9} \).”

**Part C**
Prove the induction step.

Hint: Use the fact that an \( n + 1 \) digit number may be written as the sum of an \( n \) digit number and something times \( 10^n \).

Assuming \( P(n) \), show that \( P(n+1) \). Let \( k \) be an \( n + 1 \) digit number. Then \( k = \sum_{i=0}^{n} a_i 10^i = a_n 10^n + \sum_{i=0}^{n-1} a_i 10^i \) and \( k \equiv a_n 10^n + \sum_{i=0}^{n-1} a_i 10^i \equiv a_n 10^n + \sum_{i=0}^{n-1} a_i \equiv a_n (999 \ldots 9 + 1) + \sum_{i=0}^{n-1} a_i \equiv a_n + \sum_{i=0}^{n-1} a_i \equiv \sum_{i=0}^{n} a_i \). This proves \( P(n+1) \). By induction, the hypothesis holds for all \( n \) and hence for all positive integers \( k \).

**PROBLEM 3**
The Legendre polynomial\(^1\) \( L_n(x) \) is defined recursively as follows:

\[
L_0(x) = 1 \\
L_1(x) = x \\
L_n(x) = \frac{2n-1}{n} x L_{n-1}(x) - \frac{n-1}{n} L_{n-2}(x), \quad n \geq 2
\]

Prove using generalized induction that for all \( n \geq 0 \),

\[
L_n(-1) = \begin{cases} 
1 & \text{n even}, \\
-1 & \text{n odd}.
\end{cases}
\]

Write the proposition to be proven as \( L_n(-1) = (-1)^n \).

Proposition \( P(n) \): \( L_n(-1) = (-1)^n \).

**Basis:** \( P(0) \) and \( P(1) \). \( P(0) \): \( L_0(-1) = 1 = (-1)^0 \). \( P(1) \): \( L_1(-1) = -1 = (-1)^1 \).

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\(^1\) \( L_n(x) \) is an \( n \)th degree polynomial that arises in certain problems in engineering and physics and has some useful properties. For example, it solves the differential equation \((1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0 \). Also, the set of Legendre polynomials is orthogonal in the interval \([-1,1]\) in the sense that the product of two different Legendre polynomials integrates to 0 when the integration is taken from -1 to 1, i.e., \( \int_{-1}^{1} L_m(x) L_n(x) \, dx = 0 \) for \( m \neq n \).
Induction: Assuming $P(n-2)$ and $P(n-1)$, show that $P(n)$.

\[
L_n(-1) = \frac{2n-1}{n}(-1)L_{n-1}(-1) - \frac{n-1}{n}L_{n-2}(-1)
\]

\[
= \frac{2n-1}{n}(-1)(-1)^{n-1} - \frac{n-1}{n}(-1)^{n-2}
\]

\[
= \frac{2n-1}{n}(-1)^n - \frac{n-1}{n}(-1)^n
\]

\[
= \left(\frac{2n-1}{n} - \frac{n-1}{n}\right)(-1)^n
\]

\[
= (-1)^n
\]

**PROBLEM 4**

Solve: $4S(k) - 12S(k - 1) + 9S(k - 2) = 0, \quad k \geq 3$, with $S(1) = 3, S(2) = 0$.

Characteristic equation: $4a^2 - 12a + 9 = 0$. Factor into $(2a - 3)^2 = 0$. Solutions are of the form:

\[
S(k) = (b_0 + b_1k) \left(\frac{3}{2}\right)^k.
\]

$S(1) = 3 = (b_0 + b_1) \left(\frac{3}{2}\right)$, and $S(2) = 0 = (b_0 + 2b_1) \left(\frac{3}{2}\right)^2$. Therefore $b_0 = 4, b_1 = -2$.

\[
S(k) = (4 - 2k) \left(\frac{3}{2}\right)^k.
\]

**PROBLEM 5**

**Part A**

Find a closed-form expression, i.e., give an explicit formula, for the following sequence:

$S(k) - 5S(k - 1) + 6S(k - 2) = -2$, where $S(0) = 0$ and $S(1) = 1$.

The characteristic equation for the homogeneous equation is $a^2 - 5a + 6 = 0$, therefore $a = \{2, 3\}$. This gives $S_h(k) = b_12^k + b_23^k$. For a particular solution, try $S$ is a constant, $S_p(k) = q$, which gives $q - 5q + 6q = -2$, and $q = -1$. So all possible solutions are of the form $S(k) = S_h(k) + S_p(k) = b_12^k + b_23^k - 1$. Using initial values $S(0) = b_1 + b_2 - 1 = 0$ and $S(1) = 2b_1 + 3b_2 - 1 = 1$ the system has solutions $b_1 = 1, b_2 = 0$ and

\[
S(k) = 2^k - 1.
\]

**Part B**

Find $S(6)$. 63

**Part C**

Answer the following 2 questions, recalling that $f(n) = O(g(n))$ in case there is some positive constant $C$ such that $|f(n)| \leq C \cdot |g(n)|$ for all sufficiently large values of $n$. 
Does $S(k) = O(2^k)$? Yes.

Does $S(k) = O(1)$? No.

**PROBLEM 6**

Solve the following equations for $x$ in $\mathbb{Z}(5)$, the integers $\{0, \ldots, 4\}$, where all operations are to be performed modulo 5.

1. $x^2 + 2x + 1 = 0$ Factors as $(x + 1)^2 = 0$, so $x = 4$.
2. $x^2 + 2x + 2 = 0$ Factors as $(x - 1)(x - 2) = 0$, so $x \in \{1, 2\}$.

**PROBLEM 7**

An operator $*$ on set $S$ is idempotent if $\forall a \in S, a * a = a$.

Prove that if $[S; *]$ is a group with $*$ idempotent, then $|S| = 1$.

Proof: Since $[S; *]$ is a group, there must exist an inverse for each element. $\forall a \in S, a * a = a$, and so $\forall a \in S, a^{-1} * a * a = a^{-1} * a$. Therefore $a = e$ and the only element of $S$ is the identity.

**PROBLEM 8**

Suppose that WPI has 6 parcels of land available on which to build a Dormitory, a Computer Science building, 2 parking lots, and leave 2 parcels undeveloped. In how many distinct ways can this be done? Justify your answer.

This is the problem of determining the permutations of 6 objects, where 2 pairs are indistinguishable. Solution is

$$\frac{6!}{1!1!2!2!} = \frac{6!}{4} = 180.$$ 

Think of it this way: There are 6 ways to place the Dormitory and 5 ways to place the CS building. Of the 4 parcels remaining, there are $\binom{4}{2}$ ways to place the parking lots and only a single way to leave the remaining lots undeveloped.

**PROBLEM 9**

Prove, giving a reason for each step: If $a$ and $b$ are elements of a group, then $(a * b)^{-1} = b^{-1} * a^{-1}$

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<tr>
<td>1</td>
<td>$(a * b) * (a * b)^{-1} = e$</td>
<td>definition of inverse</td>
</tr>
<tr>
<td>2</td>
<td>$a^{-1} * (a * b) * (a * b)^{-1} = a^{-1} * e$</td>
<td>multiplying by $a^{-1}$</td>
</tr>
<tr>
<td>3</td>
<td>$(e * b) * (a * b)^{-1} = a^{-1} * e$</td>
<td>associativity and definition of inverse</td>
</tr>
<tr>
<td>4</td>
<td>$b * (a * b)^{-1} = a^{-1}$</td>
<td>identity $e$</td>
</tr>
<tr>
<td>5</td>
<td>$b^{-1} * b * (a * b)^{-1} = b^{-1} a^{-1}$</td>
<td>multiplying by $b^{-1}$</td>
</tr>
<tr>
<td>6</td>
<td>$e * (a * b)^{-1} = b^{-1} a^{-1}$</td>
<td>associativity and definition of inverse</td>
</tr>
<tr>
<td>7</td>
<td>$(a * b)^{-1} = b^{-1} a^{-1}$</td>
<td>identity $e$</td>
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