Practice Examination #1 Solutions

Note: This practice examination contains more problems than would typically be found on a real examination.

**PROBLEM 1**
The Java programming language uses 64 bits to represent a double precision floating point real number. Is the set of distinct real numbers representable in this way a) finite, b) countably infinite, or c) uncountable? Explain.

With 64 bits, there can be at most $2^{64}$ distinct values represented. This constitutes a finite set.

**PROBLEM 2**
Let $f(n) = 4n^3 + n^2 + 4$. Show that $f$ is $O(n^3)$. Be sure to specify the values of $C$ and $k$ in the definition.

$f(n) \leq 4n^3 + 1n^3 + 4n^3 = 9n^3$. Therefore $f$ is $O(n^3)$ with $C = 9$ and $k = 1$.

**PROBLEM 3**
Prove or disprove

$$(p \lor q) \rightarrow r, (r \lor s) \rightarrow (t \land q) \Rightarrow (\neg q \rightarrow \neg p).$$

Give reasons for each step.

From the given propositions, we can break the compound implications into the following simpler implications:

- $p \rightarrow r$
- $q \rightarrow r$
- $r \rightarrow t$
- $r \rightarrow q$
- $s \rightarrow t$
- $s \rightarrow q$

Because $p \rightarrow r$ and $r \rightarrow q$, we have $p \rightarrow q$. The contrapositive of this is $\neg q \rightarrow \neg p$.

**PROBLEM 4**
Prove or disprove:
If $(A \cap B) \subseteq C$ then $A \subseteq C$ and $B \subseteq C$.

Give reasons for each step.

False. Consider $A = \{0\}$, $B = \{1\}$, and $C = \{2\}$. Then $(A \cap B) = \emptyset \subseteq C$, although $A \not\subseteq C$ and $B \not\subseteq C$. 
**PROBLEM 5**
Prove or disprove:
If \( A \subseteq (B - A) \) then \( A = \emptyset \).

Give reasons for each step.

Assume that \( A \) is non-empty. Let \( a \in A \). \( a \in (B - A) \), because of the \( \subseteq \) relation. \( a \in B \) and \( a \not\in A \), by definition of set difference. \( a \in A \) and \( a \not\in A \) is a contradiction. Therefore, the assumption that \( A \) is non-empty must be invalid. Therefore \( A = \emptyset \).

**PROBLEM 6**
The power set of the empty set \( \mathcal{P}(\emptyset) \) is \( \{\emptyset\} \). What is \( \mathcal{P}(\mathcal{P}(\emptyset)) \)?

\( \mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}, \{\emptyset\}, \emptyset, \{\emptyset\} \} \)

**PROBLEM 7**
Find a counterexample or prove:

\[ p \rightarrow q, \quad q \rightarrow r, \quad \neg q \Rightarrow \neg p \land \neg r \]

\( p = F, \quad q = F, \quad r = T \) is a counterexample.

**PROBLEM 8**
Find a function \( f(x) \), which is its own inverse and which has the property that \( \forall x, f(x) \neq x \). Be sure to specify the domain.

There are infinitely many such functions. For example, \( f: \mathbb{R} - \{0\} \rightarrow \mathbb{R} - \{0\}, \quad f(x) = 1/x \) is one such.

**PROBLEM 9**
For each ordered pair of integers \((a, b)\) define a function \( f_{a,b}: \mathbb{Z} \rightarrow \mathbb{Z} \) by \( f_{a,b}(n) = an + b \).

**Part A**
For which pairs \((a, b)\) is \( f_{a,b} \) one-to-one?

\( f_{a,b} \) is one-to-one as long as \( a \neq 0 \), no matter what value \( b \) has.

**Part B**
For which pairs \((a, b)\) is \( f_{a,b} \) onto?

If \( a = 1 \) or \( a = -1 \), then \( f_{a,b} \) is onto, no matter what value \( b \) has.