COMP 280 : Assignment 7 - Sample Solutions

1. $100 = 2^2 5^2$, so a number n which is relatively prime to 100 cannot be a multiple of 2 or 5. There are 50 even numbers in the set $\{1 \ldots 100\}$, so, after removing them from the set, we are left with 50 odd numbers. We have now to remove all the multiples of 5 from this set. But since we already removed all the even numbers, we just have to remove from the set the odd multiples of 5. Odd multiples of 5 less than or equal to 100 are of the form $5k$, with $k \in \{1, 3, \ldots, 19\}$, so there are 10 of them. So, once the odd multiples of 5 have been removed from the set, we are left with $50 - 10 = 40$ numbers.

2. The problem is equivalent to placing k unlabeled balls in j labeled urns, except that each urn has to contain at least one ball. So we first put one ball in each urn, and we are left with having to place $k - j$ unlabeled balls in j labeled urns, which gives us $C((k-j)+j-1, (k-j)) = C(k-1, k-j) = C(k-1, (k-1)-(k-j)) = C(k-1, j-1)$.

3. (a) Let $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Then:

$$\begin{aligned} A(x) &= \sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{+\infty} a_n x^n = 3 + 6x + \sum_{n=2}^{+\infty} (a_{n-1} + 6a_{n-2}) x^n \\ &= 3 + 6x + \sum_{n=2}^{+\infty} a_{n-1} x^n + 6 \sum_{n=2}^{+\infty} a_{n-2} x^n = 3 + 6x + x(A(x) - a_0) + 6x^2 A(x) \\ &= 3 + 3x + xA(x) + 6x^2 A(x) \end{aligned}$$

So:

$$A(x) = \frac{3+3x}{1-x-6x^2} = \frac{3+3x}{(1-3x)(1+2x)} = \frac{c_1}{1-3x} + \frac{c_2}{1+2x}$$

with $\forall x : c_1(1+2x) + c_2(1-3x) = 3 + 3x \Rightarrow c_1 + c_2 = 3$ (for $x = 0$) and $2c_1 - 3c_2 = 3$ (for $x = 1$) $\Rightarrow c_1 = 12/5$ and $c_2 = 3/5$. Then:

$$A(x) = \frac{12/5}{1-3x} + \frac{3/5}{1+2x} = \frac{12}{5} \sum_0^{+\infty} 3^n x^n + \frac{3}{5} \sum_0^{+\infty} (-2)^n x^n = \sum_0^{+\infty} \frac{3}{5} (4 * 3^n + (-2)^n) x^n$$

So:

$$a_n = \frac{3}{5}(4 * 3^n + (-2)^n)$$

(b) Let $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Then:

$$\begin{aligned} A(x) &= \sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{+\infty} a_n x^n = 4 + 10x + \sum_{n=2}^{+\infty} (6a_{n-1} - 8a_{n-2}) x^n \\ &= 4 + 10x + 6 \sum_{n=2}^{+\infty} a_{n-1} x^n - 8 \sum_{n=2}^{+\infty} a_{n-2} x^n = 4 + 10x + 6x(A(x) - a_0) - 8x^2 A(x) \\ &= 4 - 14x + 6xA(x) - 8x^2 A(x) \end{aligned}$$

So:

$$A(x) = \frac{4-14x}{1-6x+8x^2} = \frac{4-14x}{(1-2x)(1-4x)} = \frac{c_1}{1-2x} + \frac{c_2}{1-4x}$$

with $\forall x : c_1(1-4x) + c_2(1-2x) = 4 - 14x \Rightarrow c_1 + c_2 = 4$ (for $x = 0$) and $4c_1 + 2c_2 = 14$ (for $x = 1$) $\Rightarrow c_1 = 3$ and $c_2 = 1$. Then:

$$A(x) = \frac{3}{1-2x} + \frac{1}{1-4x} = 3 \sum_0^{+\infty} 2^n x^n + \sum_0^{+\infty} 4^n x^n = \sum_0^{+\infty} (3 * 2^n + 4^n) x^n$$

So:

$$a_n = 3 * 2^n + 4^n$$

4. Let $T(n)$ be the number of times the cons operation is called when applying the *dist* function to a list L of length n. $dist(x, L)$ is defined as a combination of three cons operations and $dist(x, tail(L))$. Since $dist(x, <>)$ does not use cons and since $tail(L)$ is of length $n-1$ (if L is not the empty list), we have then the following recurrence:

$$\begin{aligned} T(0) &= 0 \\ T(n) &= 3 + T(n-1) \end{aligned}$$

Then $T(n) = 3 + T(n-1) = 3 * 2 + T(n-2) = \cdots = 3i + T(n-i) = \cdots = 3n + T(n-n) = 3n$.

5. Let $S(n)$ be the set of strictly increasing sequences of positive integers that end in n and do not contain any consecutive integer, and let $a_n = |S(n)|$.

   Let $s = \langle a_1, \ldots, a_k \rangle \in S(n)$ (with $a_k = n$ by definition). Then either s contains $n-2$ or it does not.

   If it does, then it has to be of the form $\langle a_1, \ldots, n-2, a_k \rangle$, since the numbers in the sequence are in strictly increasing order. Then $\langle a_1, \ldots, n-2 \rangle$ (s with the last element removed) has to be in $S(n-2)$, since it is a strictly increasing sequence that ends in $n-2$.

   If s does not contain $n-2$, then $a_{k-1} < n-2$, and $\langle a_1, \ldots, a_{k-1}, n-1 \rangle$ (s with the last element in the sequence replaced with $n-1$) is a strictly increasing sequence which ends in $n-1$ and does not contain any consecutive integers. So $\langle a_1, \ldots, a_{k-1}, n-1 \rangle \in S(n-1)$.

   So for every element in $S(n)$ there is a corresponding element in $S(n-1)$ and / or $S(n-2)$.

   But $S(n-2) \cap S(n-1) = \emptyset$ (since the sequences in $S(n-2)$ all end in $n-2$ and the sequences in $S(n-1)$ all end in $n-1$), so for every element in $S(n)$ there is a corresponding element in $S(n-1)$ or $S(n-2)$ (exclusively). So $a_n = a_{n-1} + a_{n-2}$ ($n \geq 3$) with $a_1 = 1$ and $a_2 = 1$.