COMP 280 : Assignment 6 - Sample Solutions

1. There are $n!/(k!(n-k)!)$ ways to pick a combination of k things from $\{1, 2, \ldots, n\}$. But some of these combinations contain both 1 and 2. The combinations that contain both 1 and 2 contain 1 and 2, and $k-2$ other elements picked from $\{3, \ldots, n\}$. So the number of combinations containing both 1 and 2 is equal to the number of ways to pick a combination of $k-2$ things from $\{3, \ldots, n\}$. There are $(n-2)!/((k-2)!(n-k)!)$ such ways. So there are a total of

$$\frac{n!}{k!(n-k)!} - \frac{(n-2)!}{(k-2)!(n-k)!} = (n(n-1) - k(k-1))\frac{(n-2)!}{k!(n-k)!}$$

$$\left[= \binom{n}{k} - \binom{n-2}{k-2} = 2\binom{n-2}{k-1} + \binom{n-2}{k} \right]$$

ways to pick a combination of k things from $\{1, 2, \ldots, n\}$ without picking both 1 and 2.

2. If k is even, then there are $n^{k/2}$ ways to pick the first $k/2$ letters of the palindrome. If k is odd, then there are $n^{(k-1)/2}$ ways to pick the first $(k-1)/2$ letters, and n ways to pick the middle letter, for a total of $n^{(k+1)/2}$. So in all cases, the answer is $n^{\lceil k/2 \rceil}$.

3. Take the flags in some order, and put them on the flagpoles one by one: for the first flag you will have the choice between p possible positions (the p empty flagpoles), for the second flag you will have the choice between $p+1$ possible positions (the $p-1$ empty flagpoles, or above or below the first flag on the same flagpole), etc... In fact, each time you add a new flag, you will add another possible position for the next flag to the current number of possible positions. So the total number of ways for f flags to be put on p flagpoles is going to be $p(p+1)\cdots(p+f-1) = (p+f-1)!/(p-1)!$.

   Another way to look at it is like this: represent the different flags on the different poles as $p_1 < \cdots < p_i < f_j < \cdots < f_k < p_{i+1} < \cdots < p_p$, meaning that flags f_j and f_k are on flagpole p_i, in that order, with maybe some other flags in between. Then adding another flag is equivalent to splitting one interval into two, therefore adding one interval each time a new flag is added. Since there are initially p intervals (you cannot add something on the left of p_1), you get again $(p+f-1)!/(p-1)!$ possibilities.

   Dr. Fisler's approach: Assume you have an ordering of the flags. Then you have to decide how to distribute them across the flagpoles. This is $C(f+p-1, f)$ (this is a C2-style ball and urns problem). How many orderings of the flags are there? $f!$. So, the final solution is $f!C(f+p-1, f)$ (which, when simplified, gives the same result again).

   Another approach to Dr. Fisler's approach (from a student paper): first, think of the flags as being unlabeled (not distinct), and find all the ways they can be placed on the distinct flagpoles. This amounts to placing f unlabeled balls in p labeled urns, so there are $C(f+p-1, f)$ ways to do this. Now label the flags. Since there are $f!$ possible ways to label the flags, the product rule tells us that this will yield $f!C(f+p-1, f)$ ways to position the f flags on p flagpoles (then simplify...)

4. Given the constraints, all the strings have to be of the form $100X_1100X_2100X_3100X_4$ with four zeros left to be placed anywhere where an X_i is. So the problem is equivalent to counting the number of four-element bags that can be constructed from the set $A = \{1, 2, 3, 4\}$, where the elements of A represent positions among the different X_i. Then the result is (according to the book, formula (5.9) page 265, and the table page 263):

$$\binom{4+4-1}{4} = \binom{7}{4} = 35.$$

   Since the number of zeros and ones was given, one could also count on its fingers, if one had enough fingers...

5. Each path between n_1 and n_2 goes through between zero (direct edge) and $n-2$ (maximum number of nodes left) nodes. Each path can be represented by the list of nodes, other than n_1 and n_2, it goes through. So choosing a path that goes through p nodes is the same as picking a permutation of p nodes among the $n-2$ left ones. So there is $P(n-2,p)$ paths going through p intermediate nodes, and since any path can have between zero and $n-2$ nodes, the total number of possible paths is:

$$\sum_{i=0}^{n-2} P(n-2,i) = \sum_{i=0}^{n-2} \frac{(n-2)!}{(n-2-i)!} = (n-2)! \sum_{i=0}^{n-2} \frac{1}{(n-2-i)!} = (n-2)! \sum_{i=0}^{n-2} \frac{1}{i!}.$$

6. For each cube, there is six possible front digits, so a total of 6^4 possible front sequences. For each given front digit, there is four possible top digits, so for a given front sequence, there is 4^4 possible top sequences. So a total number of 24^4 possible front-top pairs of sequences. Now a front-top pair of sequences entirely determines an arrangement, since, knowing what the front and top digits of a cube are, one can determine what the back and bottom digits are. So there are 24^4 lists of sequences.

   But if you rotate all four cubes at the same time by 180 degrees along their vertical axis, the arrangement is not changed (only the order in the list of sequences is). The same for a rotation along the horizontal front-to-back axis. 180 degrees rotations along the horizontal left-to-right axis can be simulated by a pair of the previous two types of rotations, but 90 degrees rotations cannot, since 180 degrees rotations always flip back-front or top-bottom or left-right, but never back-top or left-front, etc... So for each arrangement, there are 2^3 possible lists of sequences (two choices for each of the three degrees of rotation - another way to see this is to just count on your fingers the number of possible lists of sequences for a single arrangement for a single cube).

   So the total number of arrangements is $24^4/2^3 = 41472$.