Closure Properties of Context-Free languages

Union

Context-free languages
are closed under: **Union**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 \cup L_2$ is context-free

Example

Language | **Grammar**

$$L_1 = \{a^n b^n\}$$

$$S_1 \to aS_1b \mid \lambda$$

$$L_2 = \{ww^R\}$$

$$S_2 \to aS_2a \mid bS_2b \mid \lambda$$

<u>Union</u>

$$L = \{a^n b^n\} \cup \{ww^R\}$$

$$S \to S_1 \mid S_2$$

In general:

For context-free languages $L_1, \; L_2$

with context-free grammars $G_1, \; G_2$

and start variables $S_1, \; S_2$

The grammar of the **union** $L_1 \cup L_2$

has new start variable S

and additional production $S \rightarrow S_1 \mid S_2$

Concatenation

Context-free languages are closed under: **Concatenation**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 L_2$ is context-free

Example

Language

$$L_1 = \{a^n b^n\}$$

$$L_2 = \{ww^R\}$$

Grammar

$$S_1 \rightarrow aS_1b \mid \lambda$$

$$S_2 \rightarrow aS_2a \mid bS_2b \mid \lambda$$

Concatenation

$$L = \{a^n b^n\}\{ww^R\}$$

$$S \rightarrow S_1 S_2$$

In general:

For context-free languages $L_1, \; L_2$
with context-free grammars $G_1, \; G_2$
and start variables $S_1, \; S_2$

The grammar of the **concatenation** $L_1 L_2$
has new start variable S
and additional production $S \to S_1 S_2$

Star Operation

Context-free languages
are closed under: **Star-operation**

L is context free ⟹ L^* is context-free

Example

Language

$$L = \{a^n b^n\}$$

Grammar

$$S \rightarrow aSb \mid \lambda$$

Star Operation

$$L = \{a^n b^n\}^*$$

$$S_1 \rightarrow SS_1 \mid \lambda$$

In general:

For context-free language L
with context-free grammar G
and start variable S

The grammar of the **star operation** L^*
has new start variable S_1
and additional production $S_1 \rightarrow SS_1 \mid \lambda$

Negative Properties of Context-Free Languages

Intersection

Context-free languages
are **not** closed under: **intersection**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 \cap L_2$

not necessarily
context-free

Example

$$L_1 = \{a^n b^n c^m\}$$

Context-free:

$$S \to AC$$

$$A \to aAb \mid \lambda$$

$$C \to cC \mid \lambda$$

$$L_2 = \{a^n b^m c^m\}$$

Context-free:

$$S \to AB$$

$$A \to aA \mid \lambda$$

$$B \to bBc \mid \lambda$$

Intersection

$L_1 \cap L_2 = \{a^n b^n c^n\}$ **NOT** context-free

Complement

Context-free languages are **not** closed under: **complement**

L is context free $\Rightarrow$ $\overline{L}$ **not** necessarily context-free

Example

$L_1 = \{a^n b^n c^m\}$ $L_2 = \{a^n b^m c^m\}$

Context-free:

$$S \to AC$$

$$A \to aAb \mid \lambda$$

$$C \to cC \mid \lambda$$

Context-free:

$$S \to AB$$

$$A \to aA \mid \lambda$$

$$B \to bBc \mid \lambda$$

Complement

$$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2 = \{a^n b^n c^n\}$$

NOT context-free

Intersection
of
Context-free languages
and
Regular Languages

The intersection of

a context-free language and

a regular language

is a context-free language

L_1 context free

L_2 regular

$\Longrightarrow$ $L_1 \cap L_2$ context-free

Machine M_1

NPDA for L_1

context-free

Machine M_2

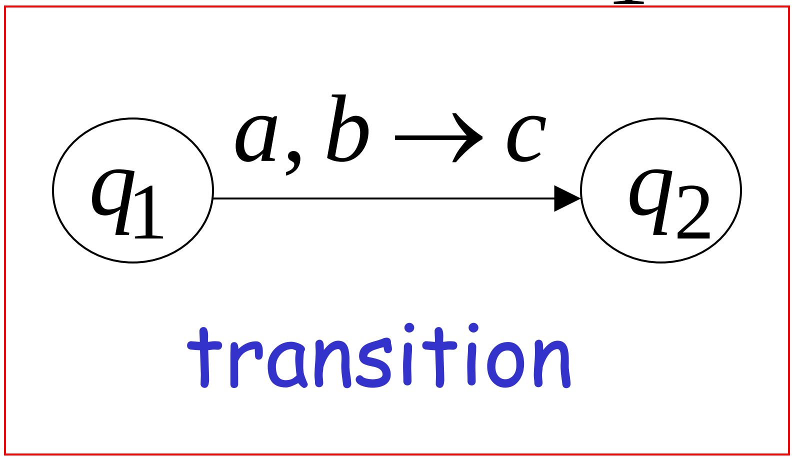
DFA for L_2

regular

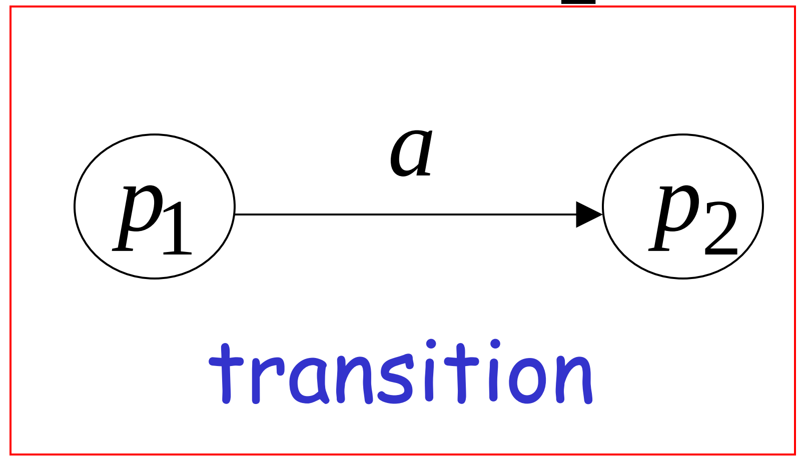
Construct a new NPDA machine M
that accepts $L_1 \cap L_2$

M simulates in parallel M_1 and M_2

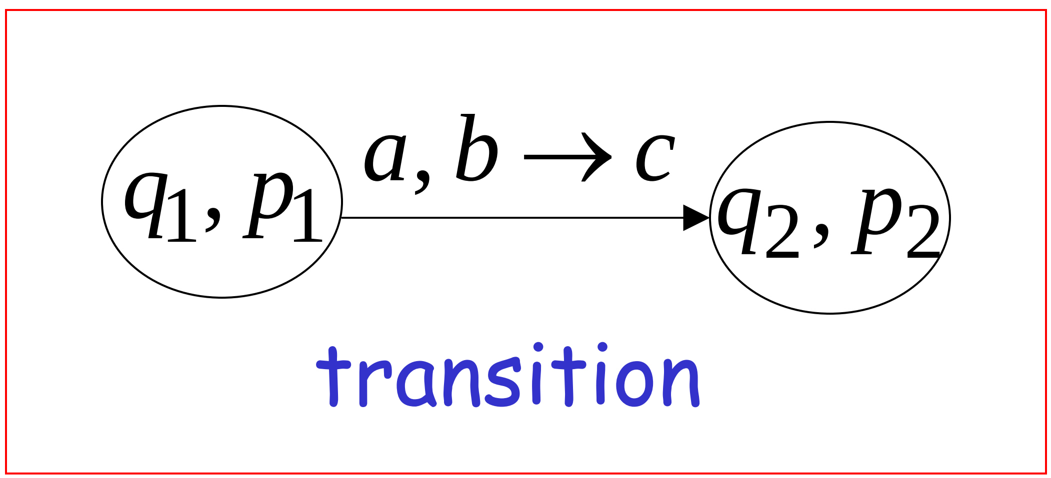
NPDA M_1

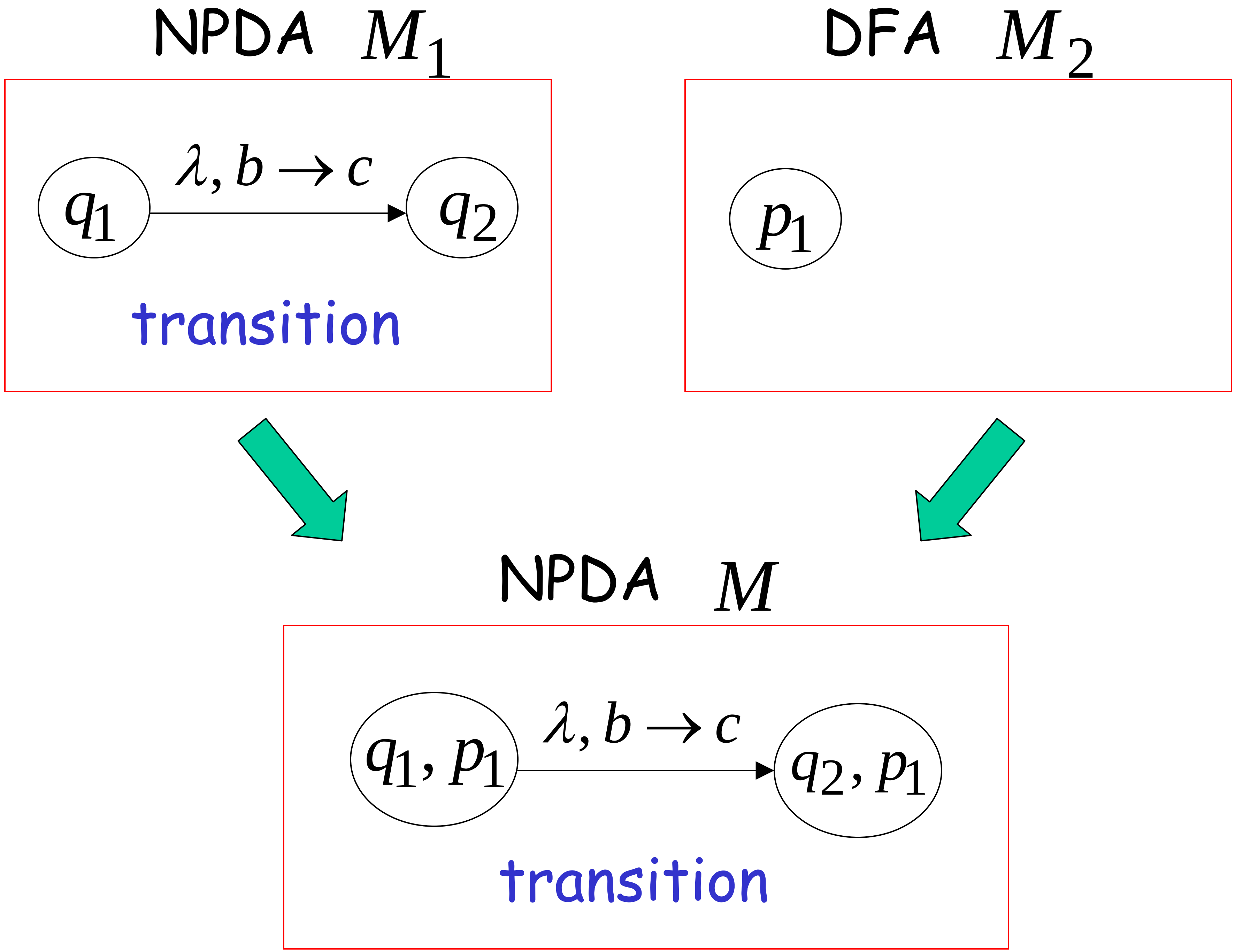


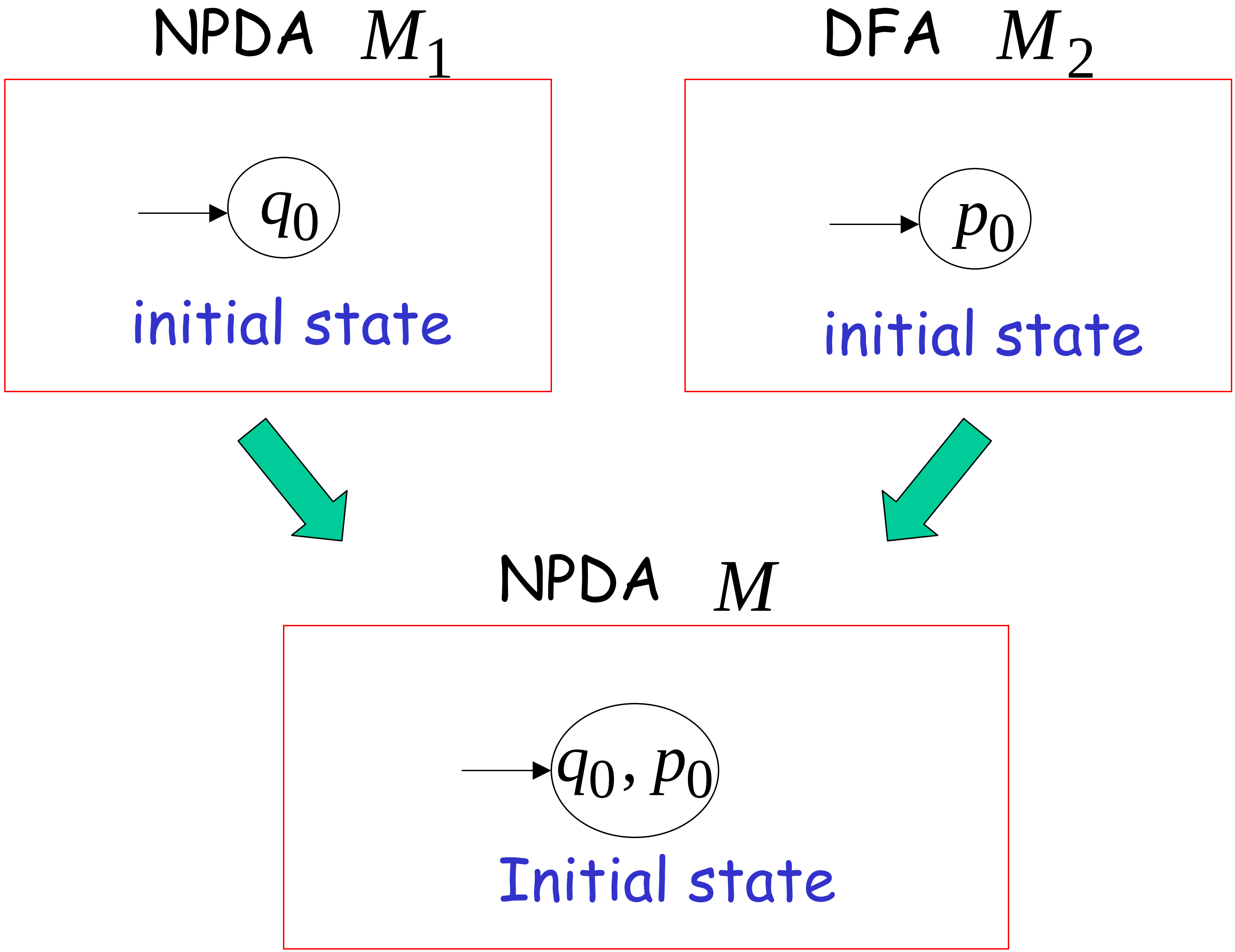
DFA M_2

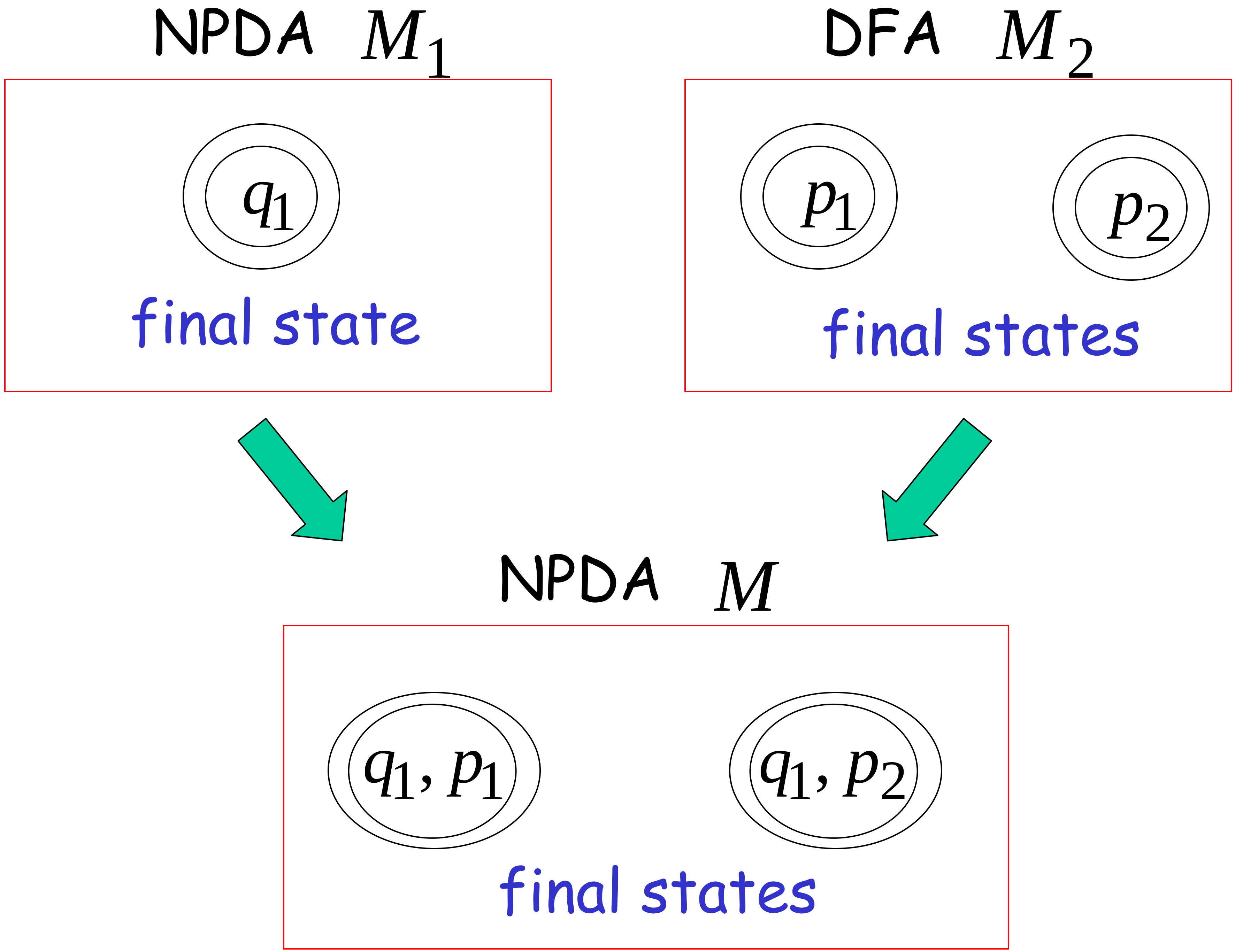


NPDA M




NPDA M_1
$\lambda, b \rightarrow c$
q_1
q_2
transition
DFA M_2
p_1
NPDA M
$\lambda, b \rightarrow c$
q_1, p_1
q_2, p_1
transition


NPDA M_1
q_0
initial state
DFA M_2
p_0
initial state
NPDA M
q_0, p_0
Initial state

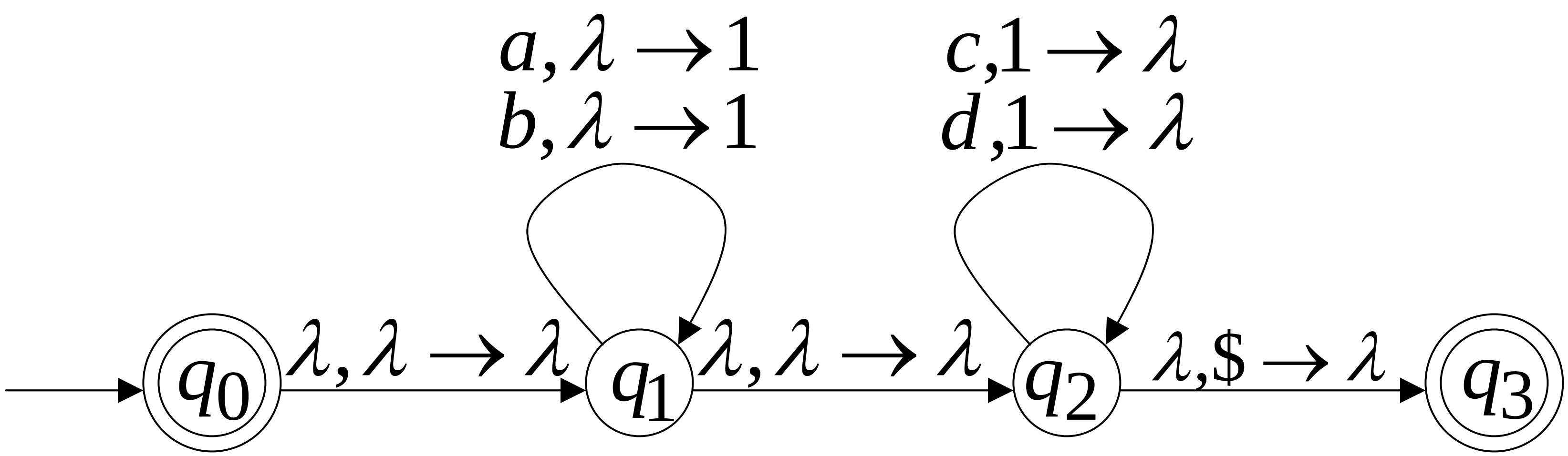

NPDA M_1
q_1
final state
DFA M_2
p_1
p_2
final states
NPDA M
q_1, p_1
q_1, p_2
final states

Example:

context-free

$$L_1 = \{w_1 w_2 : |w_1| = |w_2|, w_1 \in \{a,b\}^*, w_2 \in \{c,d\}^*\}$$

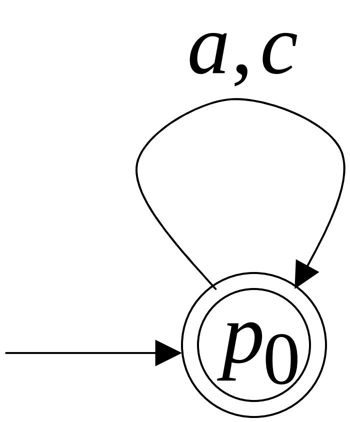
NPDA M_1



regular

$$L_2 = \{a, c\}^*$$

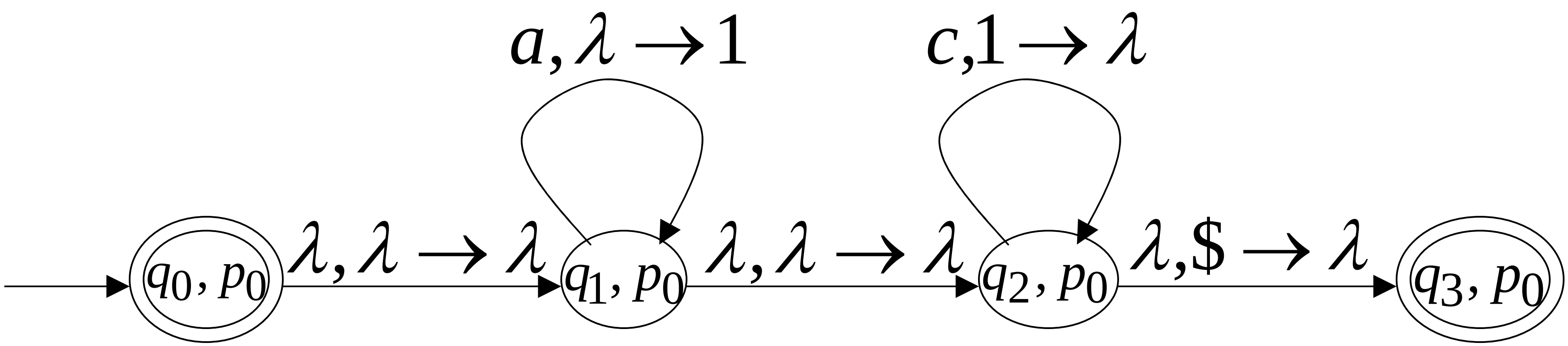
DFA M_2



context-free

Automaton for: $L_1 \cap L_2 = \{a^n c^n : n \geq 0\}$

NPDA M



In General:

M simulates in parallel M_1 and M_2

M accepts string w if and only if

M_1 accepts string w and

M_2 accepts string w

$$L(M) = L(M_1) \cap L(M_2)$$

Therefore:

M is NPDA

↓

$L(M_1) \cap L(M_2)$ is context-free

↓

$L_1 \cap L_2$ is context-free

Applications of Regular Closure

The intersection of

a context-free language and

a regular language

is a context-free language

Regular Closure

L_1 context free

L_2 regular

$\Rightarrow$ $L_1 \cap L_2$

context-free

An Application of Regular Closure

Prove that: $L = \{a^n b^n : \ n \neq 100, n \geq 0\}$

is context-free

We know:

$\{a^n b^n : n \geq 0\}$ is context-free

We also know:

$$L_1 = \{a^{100}b^{100}\}$$ is regular

$$\overline{L_1} = \{(a+b)^*\} - \{a^{100}b^{100}\}$$ is regular

$\{a^n b^n\}$ $\overline{L_1} = \{(a+b)^*\} - \{a^{100} b^{100}\}$

context-free regular

(regular closure) $\{a^n b^n\} \cap \overline{L_1}$ context-free

$$\{a^n b^n\} \cap \overline{L_1} = \{a^n b^n : \quad n \neq 100, n \geq 0\} = L$$

is context-free

Another Application of Regular Closure

Prove that: $L = \{w: \quad n_a = n_b = n_c\}$

is **not** context-free

If $L = \{w: \quad n_a = n_b = n_c\}$ is context-free

(regular closure)

Then $L \cap \{a^*b^*c^*\} = \{a^n b^n c^n\}$

context-free regular context-free

Impossible!!!

Therefore, L is **not** context free

Decidable Properties
of
Context-Free Languages

Membership Question:

for context-free grammar G

find if string $w \in L(G)$

Membership Algorithms: Parsers

- Exhaustive search parser
- **CYK** parsing algorithm

Empty Language Question:

for context-free grammar G

find if $L(G) = \emptyset$

Algorithm:

1. Remove useless variables

2. Check if start variable S is useless

Infinite Language Question:

for context-free grammar G

find if $L(G)$ is infinite

Algorithm:

1. Remove useless variables
2. Remove unit and λ productions
3. Create dependency graph for variables
4. If there is a loop in the dependency graph then the language is infinite

Example: $S \to AB$

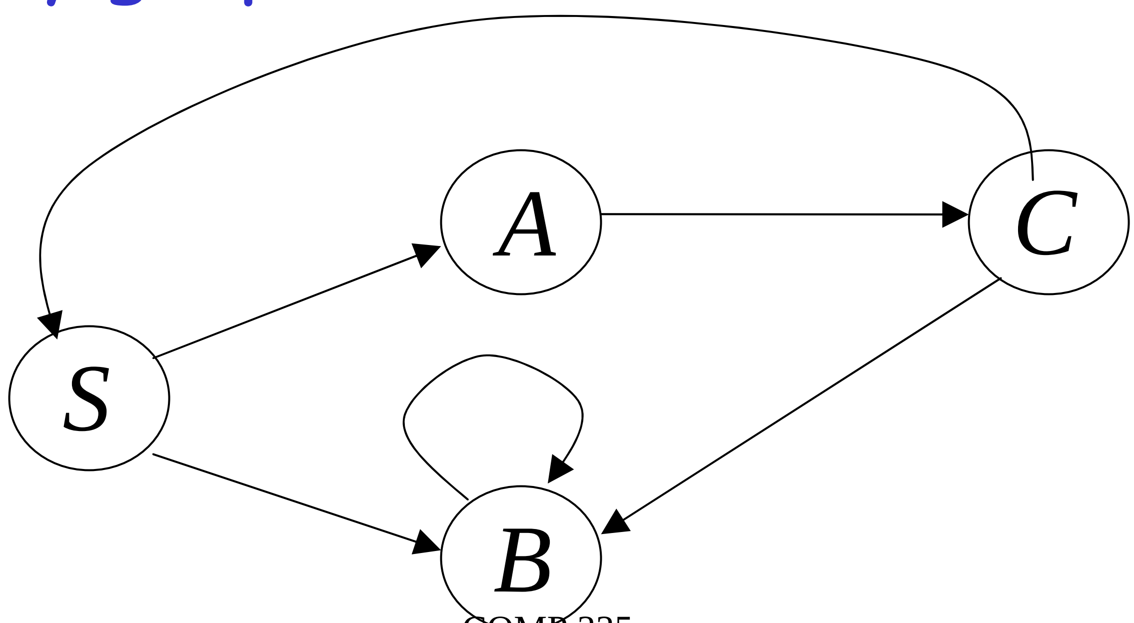
$$A \to aCb \mid a$$

$$B \to bB \mid bb$$

$$C \to cBS$$

Dependency graph

Infinite language



$$S \to AB$$

$$A \to aCb \mid a$$

$$B \to bB \mid bb$$

$$C \to cBS$$

$$S \Rightarrow AB \Rightarrow aCbB \Rightarrow acBSbB \Rightarrow acbbSbbb$$

$$S \overset{*}{\Rightarrow} acbbSbbb \overset{*}{\Rightarrow} (acbb)^2 S (bbb)^2$$

$$\overset{*}{\Rightarrow} (acbb)^i S (bbb)^i$$