Type Preservation and Normalization: Two Consequences

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Today's Goal

Theorem

Every normal derivation has the subformula property

Subformula Property

A derivation d proving $\Gamma \vdash s : \varphi$ has the subformula property iff, for every $\Delta \vdash t : \psi$ appearing in d,

either ψ is a subformula of φ,

or ψ is a subformula of χ, where some $v : \chi \in \Gamma$

Subformula Property

A derivation d proving $\Gamma \vdash s : \varphi$ has the subformula property iff, for every $\Delta \vdash t : \psi$ appearing in d,

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or ψ is a subformula of χ, where some $v : \chi \in \Gamma$

Subformula is transitive, and:

$\bot$ is a subformula of every φ

φ is a subformula of φ

φ, ψ are subformulas of:

$$\varphi \wedge \psi, \qquad \varphi \vee \psi, \qquad \varphi \rightarrow \psi$$

Today's Good

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Every normal derivation has the subformula property

Corollary

The derivation rules are consistent:
there is no derivation $\vdash s : \bot$

Proving Consistency

From the Subformula Property

Corollary

The derivation rules are consistent:
there is no derivation $\vdash s : \bot$.

Proof.

$\vdash s : \bot$ is not an instance of an axiom.

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From the Subformula Property

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The derivation rules are consistent:
there is no derivation $\vdash s\colon \bot$.

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$\vdash s\colon \bot$ is not an instance of an axiom.
It may be derived by the $\bot$ rule, if $s = \mathrm{emp}(t)$, but then there is no progress, as the premise is again of the form $\vdash t\colon \bot$.

Proving Consistency

From the Subformula Property

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The derivation rules are consistent:
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Proof.

$\vdash s \colon \bot$ is not an instance of an axiom.
It may be derived by the $\bot$ rule, if $s = \mathrm{emp}(t)$, but then there is no progress, as the premise is again of the form $\vdash t \colon \bot$.
Every other rule requires using a formula that is not a subformula of $\bot$. $\square$

Two Computational Theorems

Type Preservation and Normalization

Theorem (Type Preservation)

If $s \longrightarrow^* t$ *and* $\Gamma \vdash s : \varphi$*, then also* $\Gamma \vdash t : \varphi$*.*

Theorem (Normal Form)

If $\Gamma \vdash s : \varphi$*, then there is a normal form* t *such that* $s \longrightarrow^* t$*.*

Local reduction rules

$$
\begin{array}{lll}
\mathrm{fst}(\langle s, s' \rangle) & \longrightarrow_r & s \\
\mathrm{scd}(\langle s, s' \rangle) & \longrightarrow_r & s' \\
\mathrm{cases}(\langle \mathrm{lft}, s \rangle, t, r) & \longrightarrow_r & t\ s \\
\mathrm{cases}(\langle \mathrm{rgt}, s \rangle, t, r) & \longrightarrow_r & r\ s \\
(\lambda v\ .\ s)\ t & \longrightarrow_r & s[t/v] \qquad (\beta)
\end{array}
$$

Reducing Intro/Elim Pair: $\wedge$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ s\\ \vdots\\ \Gamma \vdash s : \varphi\end{array} \qquad \begin{array}{c}\vdots\\ t\\ \vdots\\ \Gamma \vdash t : \psi\end{array}}{\Gamma \vdash \langle s, t\rangle : \varphi \wedge \psi}}{\Gamma \vdash \mathrm{fst}(\langle s, t\rangle) : \varphi} \quad \longrightarrow \quad \begin{array}{c}\vdots\\ s\\ \vdots\\ \Gamma \vdash s : \varphi\end{array}$$

Reducing Intro/Elim Pair: $\to$

$$\dfrac{\dfrac{\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma, x : \varphi \vdash s : \psi\end{array}}{\Gamma \vdash \lambda x \,.\, s : \varphi \to \psi} \qquad \begin{array}{c}\vdots \\ t \\ \vdots \\ \Gamma \vdash t : \varphi\end{array}}{\Gamma \vdash (\lambda x \,.\, s)\; t : \psi} \quad \longrightarrow \quad \begin{array}{c}\vdots \\ s[t/x] \\ \vdots \\ \Gamma \vdash s[t/x] : \psi\end{array}$$

Compile-time reduction rules

$$
\begin{array}{lcl}
\mathrm{fst}(\mathrm{cases}(s,t,r)) & \longrightarrow_r & \mathrm{cases}(s, \mathrm{fst} \circ t, \mathrm{fst} \circ r) \\
\mathrm{scd}(\mathrm{cases}(s,t,r)) & \longrightarrow_r & \mathrm{cases}(s, \mathrm{scd} \circ t, \mathrm{scd} \circ r) \\
\mathrm{cases}(\mathrm{cases}(s,t,r),u,w) & \longrightarrow_r & \mathrm{cases}(s, \lambda v \;.\; \mathrm{cases}(t(v),u,w), \\
 & & \qquad \lambda v \;.\; \mathrm{cases}(r(v),u,w)) \\
(u\ (\mathrm{cases}(s,t,r))) & \longrightarrow_r & \mathrm{cases}(s, u \circ t, u \circ r)
\end{array}
$$

Reducing a cases/elim pair: $\wedge$

$$\dfrac{\dfrac{\Gamma \vdash s : \varphi \vee \psi \qquad \Gamma, x : \varphi \vdash t : \chi_1 \wedge \chi_2 \qquad \Gamma, y : \psi \vdash r : \chi_1 \wedge \chi_2}{\Gamma \vdash \text{cases}(s, \lambda x \,.\, t, \lambda y \,.\, r) : \chi_1 \wedge \chi_2}}{\Gamma \vdash \text{fst}(\text{cases}(s, \lambda x \,.\, t, \lambda y \,.\, r)) : \chi_1}$$

Reducing a cases/elim pair: $\wedge$

$$\dfrac{\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \Gamma, x\colon \varphi \vdash t\colon \chi_1 \wedge \chi_2 \qquad \Gamma, y\colon \psi \vdash r\colon \chi_1 \wedge \chi_2}{\Gamma \vdash \mathsf{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r)\colon \chi_1 \wedge \chi_2}}{\Gamma \vdash \mathsf{fst}(\mathsf{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r))\colon \chi_1}$$

$$\longrightarrow$$

$$\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \dfrac{\Gamma, x\colon \varphi \vdash t\colon \chi_1 \wedge \chi_2}{\Gamma, x\colon \varphi \vdash \mathsf{fst}(t)\colon \chi_1} \qquad \dfrac{\Gamma, y\colon \psi \vdash r\colon \chi_1 \wedge \chi_2}{\Gamma, y\colon \varphi \vdash \mathsf{fst}(r)\colon \chi_1}}{\Gamma \vdash \mathsf{cases}(s, \lambda x\,.\,\mathsf{fst}(t), \lambda y\,.\,\mathsf{fst}(r))\colon \chi_1}$$

Reducing a cases/elim pair: $\vee$

$$\dfrac{\dfrac{\Gamma \vdash s : \varphi \vee \psi \qquad \begin{matrix} \Gamma, x : \varphi \vdash t : \chi_1 \vee \chi_2 \\ \Gamma, y : \psi \vdash r : \chi_1 \vee \chi_2 \end{matrix}}{\Gamma \vdash \text{cases}(s, \lambda x\, .\, t, \lambda y\, .\, r) : \chi_1 \vee \chi_2} \qquad \begin{matrix} \Gamma, z : \chi_1 \vdash u : \rho \\ \Gamma, v : \chi_2 \vdash w : \rho \end{matrix}}{\Gamma \vdash f : \rho}$$

Reducing a cases/elim pair: $\vee$

Omitted: Analogous deriv. of $\Gamma, y\colon \psi \vdash g_2\colon \rho$

$$\dfrac{\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \begin{array}{c}\Gamma, x\colon \varphi \vdash t\colon \chi_1 \vee \chi_2 \\ \Gamma, y\colon \psi \vdash r\colon \chi_1 \vee \chi_2\end{array}}{\Gamma \vdash \mathrm{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r)\colon \chi_1 \vee \chi_2} \qquad \begin{array}{c}\Gamma, z\colon \chi_1 \vdash u\colon \rho \\ \Gamma, v\colon \chi_2 \vdash w\colon \rho\end{array}}{\Gamma \vdash f\colon \rho}$$

$$\longrightarrow$$

$$\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \dfrac{\Gamma, x\colon \varphi \vdash t\colon \chi_1 \vee \chi_2 \qquad \begin{array}{c}\Gamma, x\colon \varphi, z\colon \chi_1 \vdash u\colon \rho \\ \Gamma, x\colon \varphi, v\colon \chi_2 \vdash w\colon \rho\end{array}}{\Gamma, x\colon \varphi \vdash g_1\colon \rho}}{\Gamma \vdash f'\colon \rho}$$

Reducing a cases/elim pair: $\lor$

Omitted: Analogous deriv. of $\Gamma, y : \psi \vdash g_2 : \rho$

$$\dfrac{\dfrac{\Gamma \vdash s : \varphi \lor \psi \qquad \begin{array}{c}\Gamma, x : \varphi \vdash t : \chi_1 \lor \chi_2 \\ \Gamma, y : \psi \vdash r : \chi_1 \lor \chi_2\end{array}}{\Gamma \vdash \text{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r) : \chi_1 \lor \chi_2} \qquad \begin{array}{c}\Gamma, z : \chi_1 \vdash u : \rho \\ \Gamma, v : \chi_2 \vdash w : \rho\end{array}}{\Gamma \vdash f : \rho}$$

$$\longrightarrow$$

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$$f = \text{cases}(\text{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r), \lambda z\,.\,u, \lambda v\,.\,w)$$

$$g_1 = \text{cases}(t, \lambda z\,.\,u, \lambda v\,.\,w) \quad g_2 = \text{cases}(r, \lambda z\,.\,u, \lambda v\,.\,w)$$

$$f' = \text{cases}(u, g_1, g_2)$$

The Reduction Relation

$$\frac{s \longrightarrow_r t}{s \longrightarrow t} \qquad \frac{s \longrightarrow t}{\mathcal{C}[s] \longrightarrow \mathcal{C}[t]}$$

$$\frac{}{s \longrightarrow^* s} \qquad \frac{s \longrightarrow^* t \qquad t \longrightarrow u}{s \longrightarrow^* u}$$

Contexts $\mathcal{C}[x]$

Replace any s, t, u with an x to make a context $\mathcal{C}[x]$:

$$
\begin{array}{rclclcl}
\mathcal{C}[x] & ::= & x & \mid & & & \\
 & & \langle \mathcal{C}'[x], t \rangle & \mid & \langle s, \mathcal{C}'[x] \rangle & \mid & \\
 & & \mathsf{fst}(\mathcal{C}'[x]) & \mid & \mathsf{scd}(\mathcal{C}'[x]) & \mid & \\
 & & (\lambda v \,.\, \mathcal{C}'[x]) & \mid & (\mathcal{C}'[x]\ t) & \mid & (s\ \mathcal{C}'[x]) \\
 & & \langle \mathsf{lft}, \mathcal{C}'[x] \rangle & \mid & \langle \mathsf{rgt}, \mathcal{C}'[x] \rangle & \mid & \\
 & & \mathsf{cases}(\mathcal{C}'[x], t, r) & \mid & \mathsf{cases}(s, \mathcal{C}'[x], r) & \mid & \mathsf{cases}(s, t, \mathcal{C}'[x])
\end{array}
$$

Two Computational Theorems

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If $s \longrightarrow^* t$ *and* $\Gamma \vdash s : \varphi$, *then also* $\Gamma \vdash t : \varphi$.

Theorem (Normal Form)

If $\Gamma \vdash s : \varphi$, *then there is a normal form* t *such that* $s \longrightarrow^* t$.

A Corollary: Normal Derivations

Corollary

1. *If φ is derivable from Γ, then there is a normal derivation t such that $\Gamma \vdash t : \varphi$*
2. *If additionally $\Gamma = \emptyset$, then t is closed (i.e. no free variables)*

A Corollary: Normal Derivations

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1. *If φ is derivable from Γ, then there is a normal derivation t such that $\Gamma \vdash t : \varphi$*
2. *If additionally $\Gamma = \emptyset$, then t is closed (i.e. no free variables)*

Proof.

1. If φ is derivable from Γ, then for some s, $\Gamma \vdash s : \varphi$.
By normal form, $s \longrightarrow t$ for some normal t.
By type preservation, $\Gamma \vdash t : \varphi$.

□

A Corollary: Normal Derivations

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1. If φ is derivable from Γ, then for some s, $\Gamma \vdash s : \varphi$.
By normal form, $s \longrightarrow t$ for some normal t.
By type preservation, $\Gamma \vdash t : \varphi$.

2. Via the Context Lemma, which says:

$$\text{If } \Gamma \vdash s : \varphi, \text{ then } \mathrm{fv}(s) \subseteq \mathrm{dom}(\Gamma).$$

□

A Normal Proof

$$
\dfrac{\dfrac{\dfrac{p,\ (p \to \bot) \wedge q \ \vdash\ (p \to \bot) \wedge q}{p,\ (p \to \bot) \wedge q \ \vdash\ p \to \bot} \qquad p,\ (p \to \bot) \wedge q \ \vdash\ p}{\dfrac{p,\ (p \to \bot) \wedge q \ \vdash\ \bot}{p,\ (p \to \bot) \wedge q \ \vdash\ q}}}{\dfrac{(p \to \bot) \wedge q \ \vdash\ p \to q}{\vdash\ ((p \to \bot) \wedge q) \to (p \to q)}}
$$

$$\lambda x \ .\ \lambda y \ .\ \mathrm{emp}(\mathrm{fst}(x)\ y)$$

Another Normal Proof

$$
\dfrac{p, (p \lor q) \to r \ \vdash \ (p \lor q) \to r \qquad \dfrac{p, (p \lor q) \to r \ \vdash \ p}{p, (p \lor q) \to r \ \vdash \ p \lor q}}{\dfrac{\dfrac{p, (p \lor q) \to r \ \vdash \ r}{(p \lor q) \to r \ \vdash \ p \to r}}{\vdash \ ((p \lor q) \to r) \to (p \to r)}}
$$

$$\lambda y \,.\, \lambda x \,.\, (y \ \langle \mathsf{fst}, x \rangle)$$

Normal Derivations, 1

Regarding s as a tree with the conclusion at the root

If d is a normal derivation, then

working upward from any point through *major premises*,
every application of an introduction rule
is reached before
any application of an elimination rule

Normal Derivations, 1

Regarding *s* as a tree with the conclusion at the root

If *d* is a normal derivation, then

working upward from any point through *major premises*,
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All premises in an introduction rule are major

The major premise of an elimination rule is
the premise containing the connective to be eliminated

Major and Minor Premises: Conjunction

$$\frac{\Gamma \vdash s : \varphi \qquad \Gamma \vdash t : \psi}{\Gamma \vdash \langle s, t \rangle : \varphi \wedge \psi}$$

$$\frac{\Gamma \vdash s : \varphi \wedge \psi}{\Gamma \vdash \mathsf{fst}(s) : \varphi} \qquad \frac{\Gamma \vdash s : \varphi \wedge \psi}{\Gamma \vdash \mathsf{scd}(s) : \psi}$$

Major and Minor Premises: Implication

$$\frac{\Gamma, x\colon \varphi \ \vdash \ s\colon \psi}{\Gamma \ \vdash \ \lambda x\,.\,s\colon \varphi \to \psi}$$

$$\frac{\Gamma \ \vdash \ s\colon \varphi \to \psi \qquad \Gamma \ \vdash \ t\colon \varphi}{\Gamma \ \vdash \ (s\ t)\colon \psi}$$

Major and Minor Premises: Disjunction

$$\frac{\Gamma \vdash s : \varphi}{\Gamma \vdash \langle \mathsf{lft}, s \rangle : \varphi \vee \psi} \qquad \frac{\Gamma \vdash s : \psi}{\Gamma \vdash \langle \mathsf{rgt}, s \rangle : \varphi \vee \psi}$$

$$\frac{\Gamma \vdash s : \varphi \vee \psi \qquad \Gamma, x : \varphi \vdash t : \chi \qquad \Gamma, y : \psi \vdash r : \chi}{\Gamma \vdash \mathsf{cases}(s, \lambda x \,.\, t, \lambda y \,.\, r) : \chi}$$

Major and Minor Premises: Axiom and Falsehood

$$\frac{}{\Gamma, x\colon \varphi \vdash x\colon \varphi} \qquad \frac{\Gamma \vdash x\colon \bot}{\Gamma \vdash \mathrm{emp}(x)\colon \varphi}$$

Normal Derivations, 1

Regarding s as a tree with the conclusion at the root

If d is a normal derivation, then

working upward from any point through *major premises*,
every application of an introduction rule
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Normal Derivations, 2

If d is a normal derivation, and p is any upwards path in d

if p traverses only elimination rules

and p traverses a disjunction elimination inference

then it is below any other elimination rule

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If d is a normal derivation, and p is any upwards path in d

if p traverses only elimination rules

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By the compile-time rules

Reducing a cases/elim pair: $\wedge$

$$\dfrac{\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \Gamma, x\colon \varphi \vdash t\colon \chi_1 \wedge \chi_2 \qquad \Gamma, y\colon \psi \vdash r\colon \chi_1 \wedge \chi_2}{\Gamma \vdash \mathsf{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r)\colon \chi_1 \wedge \chi_2}}{\Gamma \vdash \mathsf{fst}(\mathsf{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r))\colon \chi_1}$$

$$\longrightarrow$$

$$\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \dfrac{\Gamma, x\colon \varphi \vdash t\colon \chi_1 \wedge \chi_2}{\Gamma, x\colon \varphi \vdash \mathsf{fst}(t)\colon \chi_1} \qquad \dfrac{\Gamma, y\colon \psi \vdash r\colon \chi_1 \wedge \chi_2}{\Gamma, y\colon \varphi \vdash \mathsf{fst}(r)\colon \chi_1}}{\Gamma \vdash \mathsf{cases}(s, \lambda x\,.\,\mathsf{fst}(t), \lambda y\,.\,\mathsf{fst}(r))\colon \chi_1}$$

Normal Derivations, 3

If d is a normal derivation, and p is any upwards path in d

if p traverses only elimination rules

then p traverses at most one major premise of a disjunction elimination inference

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If d is a normal derivation, and p is any upwards path in d

if p traverses only elimination rules

then p traverses at most one major premise of a disjunction elimination inference

By the case/elim rule for $\vee$
and the previous claim

Reducing a cases/elim pair: $\vee$

Omitted: Analogous deriv. of $\Gamma, y : \psi \vdash g_2 : \rho$

$$\dfrac{\dfrac{\Gamma \vdash s : \varphi \vee \psi \qquad \begin{array}{c}\Gamma, x : \varphi \vdash t : \chi_1 \vee \chi_2 \\ \Gamma, y : \psi \vdash r : \chi_1 \vee \chi_2\end{array}}{\Gamma \vdash \mathrm{cases}(s, \lambda x \, . \, t, \lambda y \, . \, r) : \chi_1 \vee \chi_2} \qquad \begin{array}{c}\Gamma, z : \chi_1 \vdash u : \rho \\ \Gamma, v : \chi_2 \vdash w : \rho\end{array}}{\Gamma \vdash f : \rho}$$

$$\longrightarrow$$

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$$f = \mathrm{cases}(\mathrm{cases}(s, \lambda x \, . \, t, \lambda y \, . \, r), \lambda z \, . \, u, \lambda v \, . \, w)$$

$$g_1 = \mathrm{cases}(t, \lambda z \, . \, u, \lambda v \, . \, w) \quad g_2 = \mathrm{cases}(r, \lambda z \, . \, u, \lambda v \, . \, w)$$

$$f' = \mathrm{cases}(u, g_1, g_2)$$

Normal Derivations, 4

If d is a normal derivation, and p is any upwards path in d

if p traverses only introduction rules

then each successive right hand side is a subformula of the one below it

By the form of the introduction rules

Today's Goal

Theorem

Every normal derivation has the subformula property

Why it's true

Theorem

Every normal derivation d of $\Gamma \vdash s : \varphi$ has the subformula property

Proof.

1. Conclusion of an introduction rule: subformula of φ

□

Why it's true

Theorem

Every normal derivation d of $\Gamma \vdash s : \varphi$ has the subformula property

Proof.

1. Conclusion of an introduction rule: subformula of φ
2. Major premise of an elimination rule: subformula of some ψ in Γ

□

Why it's true

Theorem

Every normal derivation d of $\Gamma \vdash s : \varphi$ has the subformula property

Proof.

1. Conclusion of an introduction rule: subformula of φ
2. Major premise of an elimination rule: subformula of some ψ in Γ
3. Minor premise of $\rightarrow$-elimination: subformula of the major premise

□

Why it's true

Theorem

Every normal derivation d of $\Gamma \vdash s : \varphi$ *has the subformula property*

Proof.

1. Conclusion of an introduction rule: subformula of φ
2. Major premise of an elimination rule: subformula of some ψ in Γ
3. Minor premise of $\rightarrow$-elimination: subformula of the major premise
4. Minor premise of $\vee$-elimination: subformula of φ

□

Today's Goal

Theorem

Every normal derivation has the subformula property

Corollary

The derivation rules are consistent:
there is no derivation $\vdash s : \bot$