

How to Prove Properties by Induction on Formulas

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Formulas are finite, recursively generated objects, and therefore we can prove properties about all formulas using *structural induction*. “Structural induction” refers to induction principles that work directly on the structure of recursively generated objects, rather than using natural numbers to measure properties like the height or length of an object. In this note, we will:

- Give the recursive definition of formulas, omitting the implication operator \Rightarrow for simplicity. Since $\alpha \Rightarrow \beta$ is equivalent to $\neg\alpha \vee \beta$, nothing is lost here.
- Formalize the structural induction principle for formulas that follows from the definition.
- Give an example of the use of the induction principle to prove a property of all formulas.

In particular, we prove this. Suppose that S is a set of atoms, meaning that $S \subseteq \mathcal{L}$ for a language \mathcal{L} . If a formula α uses only atoms in S , and two models \mathbb{M} and \mathbb{M}' agree on all the atoms in S , then:

$$\mathbb{M} \models \alpha \quad \text{if and only if} \quad \mathbb{M}' \models \alpha.$$

That is, whether a model satisfies a formula depends only on what the values it assigns to atoms that appear in the formula. It’s just intended to illustrate how to prove properties of formulas by induction.

Definition 1 (Formulas of Propositional Logic) Let \mathcal{L} be a set; we will refer to the members of \mathcal{L} as *atoms* or *propositional letters*. The set \mathcal{F} of (propositional logic) *formulas over \mathcal{L}* is defined recursively by:

1. If $A \in \mathcal{L}$, then A is a formula over \mathcal{L} ;
2. If β and γ are formulas over \mathcal{L} , then so are:

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- (a) $\neg\beta$;
- (b) $\beta \vee \gamma$;
- (c) $\beta \wedge \gamma$. ///

The induction principle for formulas over \mathcal{L} then says:

Proposition 1 (Structural Induction over Formulas) Suppose that $\text{Prop}(\alpha)$ is a property of formulas α . If Clauses 1 and 2 are true, then $\text{Prop}(\alpha)$ holds of *all* formulas over \mathcal{L} :

1. If $A \in \mathcal{L}$, then $\text{Prop}(A)$ is true; and
2. Let β and γ be formulas over \mathcal{L} , and suppose that

$$\text{Prop}(\beta) \quad \text{and} \quad \text{Prop}(\gamma) \quad \dagger$$

are true. Then:

- (a) $\text{Prop}(\neg\beta)$;
- (b) $\text{Prop}(\beta \vee \gamma)$;
- (c) $\text{Prop}(\beta \wedge \gamma)$. ///

Clause 1 is the *base case*. Clause 2 is the *induction step*, and its assumption (\dagger) is the *induction hypothesis*. Generally speaking, the base case is easy, and the most important things in writing a convincing proof by induction are to state the induction hypothesis clearly, and to show why the property is preserved. In our case of proofs about formulas, that means showing how to get from the induction hypothesis to the conclusions 2(a)–(c).

For our example, we need to say what we mean by a “formula uses only atoms in S .” We define it this way:

Definition 2 Formula α uses *only atoms in S* iff:

- α is an atom A , and $A \in S$; or
- α is $\neg\beta$, and β uses only atoms in S ; or
- α is $\beta \vee \gamma$ or $\beta \wedge \gamma$, and β and γ use only atoms in S .

So for instance, $(A \vee B) \wedge (A \wedge \neg D)$ uses only atoms in $\{A, B, C, D\}$. It doesn’t have to use all of them, it just has to select only from the set.

So here’s our example induction.

Proposition 2 Suppose that S is a set of atoms, and $\mathbb{M}(A) = \mathbb{M}'(A)$ for every $X \in S$. For all formulas α , if α uses only atoms in S , then

$$\mathbb{M} \models \alpha \quad \text{if and only if} \quad \mathbb{M}' \models \alpha.$$

Proof. Consider a set of atoms S , and a pair of models \mathbb{M}, \mathbb{M}' such that $\mathbb{M}(A) = \mathbb{M}'(A)$ for every A in S .

Base Case. Let α be an atom $A \in \mathcal{L}$. By the definition of “uses only,” $A \in S$. Thus, $\mathbb{M}(A) = \mathbb{M}'(A)$. By the definition of \models for atoms, we only need to show that $\mathbb{M}(A) = 1$ if and only if $\mathbb{M}'(A) = 1$. But this follows immediately from $\mathbb{M}(A) = \mathbb{M}'(A)$.

Induction Step. Suppose as our induction hypothesis (our instance of †):

- If β uses only atoms in S , then $\mathbb{M} \models \beta$ if and only if $\mathbb{M}' \models \beta$, and
- If γ uses only atoms in S , then $\mathbb{M} \models \gamma$ if and only if $\mathbb{M}' \models \gamma$.

We must show that the property still holds of $\neg\beta$, $\beta \vee \gamma$, and $\beta \wedge \gamma$.

$\neg\beta$: Either (i) β uses only atoms in S , or (ii) it is not the case that β uses only atoms in S . In case (i), the induction hypothesis implies $\mathbb{M} \models \beta$ if and only if $\mathbb{M}' \models \beta$. So $\mathbb{M} \not\models \beta$ iff $\mathbb{M}' \not\models \beta$. Hence, by the definition of \models , $\mathbb{M} \models \neg\beta$ if and only if $\mathbb{M}' \models \neg\beta$.

In case (ii), by the definition, it is not the case that $\neg\beta$ uses only atoms in S . Thus, any implication “If $\neg\beta$ uses only atoms in S , then ...” is definitely true.

$\beta \vee \gamma$: Either (i) both β and γ use only atoms in S , or (ii) it is not the case that both β and γ use only atoms in S .

In case (i), the induction hypothesis implies $\mathbb{M} \models \beta$ if and only if $\mathbb{M}' \models \beta$, and likewise for γ . Hence, by the definition of \models , $\mathbb{M} \models \beta \vee \gamma$ if and only if $\mathbb{M}' \models \beta \vee \gamma$.

In case (ii), by the definition, it is not the case that $\beta \vee \gamma$ uses only atoms in S . Thus, any implication “If $\beta \vee \gamma$ uses only atoms in S , then ...” is definitely true.

$\beta \wedge \gamma$: Just like the previous case. ///

A few comments may be helpful. First, the propositional logic meaning of *implies* is crucial for making this proof work in case (ii) for each of the connectives. As soon as the hypothesis is false, the truth of the implication “comes for free.”

Second, in the induction, I’ve tried to make it clear what changes and what remains the same. In particular, the set S and the models \mathbb{M} and \mathbb{M}' stay the same from the beginning of the proof to the end. The parameter that changes is the formula α , and of course its components β, γ .

Third, in writing a proof by induction, you want to make very clear what assumptions you’re making at each point. In particular, what the induction hypothesis is should be very explicit, and each time you use it make clear how it’s paying off.

The structure of the clauses 1 and 2 in the induction principle exactly match the recursive definition of the set of formulas. A recursive definition always justifies a matching induction principle in this way.