Consequence Relations and Natural Deduction

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31 Aug 2010

Consequence Relation

Definition

$\preceq$: relates finite sets of formulas Γ and individual formulas φ

$\preceq$ is a *consequence relation* iff

Reflexivity: $\Gamma, \varphi \preceq \varphi$

Transitivity: $\Gamma \preceq \psi$ if $\Gamma \preceq \varphi$ and $\Gamma, \varphi \preceq \psi$

Weakening: $\Gamma, \Delta \preceq \varphi$ if $\Gamma \preceq \varphi$

Substitution: $\Gamma[t_1/x_1, \ldots t_n/x_n] \preceq \varphi[t_1/x_1, \ldots t_n/x_n]$ if $\Gamma \preceq \varphi$

Consequence as an Ordering

Consequence and Entailment

$\Gamma \Vdash \varphi$, holds iff, for all models $\mathbb{M}$:

If for each $\psi \in \Gamma$, $\mathbb{M} \models \psi$,

then $\mathbb{M} \models \varphi$.

$\Vdash$ is a Consequence Relation

Reflexivity: $\Gamma, \varphi \Vdash \varphi$;

Transitivity: $\Gamma \Vdash \varphi$ and $\Gamma, \varphi \Vdash \psi$ imply $\Gamma \Vdash \psi$; and

Weakening: $\Gamma \Vdash \varphi$ implies $\Gamma, \Delta \Vdash \varphi$.

Natural Deduction

A *natural deduction derivation* is a tree in which each judgment is the conclusion of a rule.

The conclusion, the root, goes at the bottom

Conjunction Rules

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}$$

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

Implication Rules

$$\frac{\Gamma\varphi \;\vdash\; \psi}{\Gamma \;\vdash\; \varphi \to \psi}$$

$$\frac{\Gamma \;\vdash\; \varphi \to \psi \qquad \Gamma \;\vdash\; \varphi}{\Gamma \;\vdash\; \psi}$$

Disjunction Rules

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

$$\frac{\Gamma \vdash \varphi \vee \psi \qquad \Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma \vdash \chi}$$

Axiom and Bottom

$$\frac{}{\Gamma, \varphi \vdash \varphi} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

A Derivation

$$\dfrac{\dfrac{\dfrac{}{p \wedge q \;\vdash\; p \wedge q}}{p \wedge q \;\vdash\; p}}{\dfrac{p \wedge q \;\vdash\; p \vee q}{\vdash\; (p \wedge q) \rightarrow (p \vee q)}}$$

Lemma: Derivable Judgments form a Consequence Relation

Proof: 1

1. **Reflexivity** holds because $\dfrac{}{\Gamma, \varphi \vdash \varphi}$ is always a derivation.
2. **Transitivity** holds by:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ d_1\\ \vdots\\ \Gamma, \varphi \vdash \psi\end{array}}{\Gamma \vdash \varphi \to \psi} \qquad \begin{array}{c}\vdots\\ d_2\\ \vdots\\ \Gamma \vdash \varphi\end{array}}{\Gamma \vdash \psi}$$

Lemma: Derivable Judgments form a Consequence Relation

Proof: 2

3. Weakening holds by *induction on derivations*:

Base Case Suppose that there is a derivation of $\Gamma \vdash \varphi$ consisting only of an application of the Axiom rule. That is, $\varphi \in \Gamma$. Thus, $\varphi \in \Gamma, \Delta$, so $\dfrac{}{\Gamma, \Delta \vdash \varphi}$ is an application of the Axiom rule.

Lemma: Derivable Judgments form a Consequence Relation

Proof: 3

Induction Step Suppose that we are given a derivation d where the last step is an application of one of the rules, and the previous steps generate one or more subderivations d_i, each with conclusion $\Gamma_i \vdash \psi_i$.
Induction hypothesis. Assume that for each of the subderivations d_i, there is a weakened subderivation $W(d_i)$ such that $W(d_i)$ has conclusion $\Gamma_i, \Delta \vdash \psi_i$.
Construct the desired derivation of $\Gamma, \Delta \vdash \varphi$ by combining the weakened subderivations $W(d_i)$ using the *same* rule of inference.

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Construct the desired derivation of $\Gamma, \Delta \vdash \varphi$ by combining the weakened subderivations $W(d_i)$ using the *same* rule of inference.

This is really a Program operating on Proofs