Type Preservation and Normalization: Two Consequences

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Today's Goal

Theorem

Every normal derivation has the subformula property

Subformula Property

A derivation d proving $\Gamma \vdash s\colon \varphi$ has the subformula property iff, for every $\Delta \vdash t\colon \psi$ appearing in d,

either ψ is a subformula of φ,

or ψ is a subformula of χ, where some $v\colon \chi \in \Gamma$

Subformula is transitive, and:

$\bot$ is a subformula of every φ

φ is a subformula of φ

φ, ψ are subformulas of:

$$\varphi \wedge \psi, \qquad \varphi \vee \psi, \qquad \varphi \to \psi$$

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Theorem

Every normal derivation has the subformula property

Corollary

The derivation rules are consistent: there is no derivation $\vdash s\colon \bot$

Proving Consistency

From the Subformula Property

Corollary

The derivation rules are consistent: there is no derivation $\vdash s\colon \bot$.

Proof.

$\vdash s\colon \bot$ is not an instance of an axiom.
It may be derived by the $\bot$ rule, if $s = \text{emp}(t)$, but then there is no progress, as the premise is again of the form $\vdash t\colon \bot$.
Every other rule requires using a formula that is not a subformula of $\bot$. □

Two Computational Theorems

Type Preservation and Normalization

Theorem (Type Preservation)

If $s \longrightarrow^* t$ and $\Gamma \vdash s\colon \varphi$, then also $\Gamma \vdash t\colon \varphi$.

Theorem (Normal Form)

If $\Gamma \vdash s\colon \varphi$, then there is a normal form t such that $s \longrightarrow^* t$.

Local reduction rules

$$
\begin{array}{lll}
\mathsf{fst}(\langle s, s' \rangle) & \longrightarrow_r & s \\
\mathsf{scd}(\langle s, s' \rangle) & \longrightarrow_r & s' \\
\mathsf{cases}(\langle \mathsf{lft}, s \rangle, t, r) & \longrightarrow_r & t\ s \\
\mathsf{cases}(\langle \mathsf{rgt}, s \rangle, t, r) & \longrightarrow_r & r\ s \\
(\lambda v\ .\ s)\ t & \longrightarrow_r & s[t/v] \qquad (\beta)
\end{array}
$$

Reducing Intro/Elim Pair: $\wedge$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma \vdash s \colon \varphi\end{array} \qquad \begin{array}{c}\vdots \\ t \\ \vdots \\ \Gamma \vdash t \colon \psi\end{array}}{\Gamma \vdash \langle s, t \rangle \colon \varphi \wedge \psi}}{\Gamma \vdash \mathsf{fst}(\langle s, t \rangle) \colon \varphi}
\quad \longrightarrow \quad
\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma \vdash s \colon \varphi\end{array}
$$

Reducing Intro/Elim Pair: $\rightarrow$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots \\ s \\ \vdots \\ \Gamma, x \colon \varphi \vdash s \colon \psi\end{array}}{\Gamma \vdash \lambda x\ .\ s \colon \varphi \rightarrow \psi} \qquad \begin{array}{c}\vdots \\ t \\ \vdots \\ \Gamma \vdash t \colon \varphi\end{array}}{\Gamma \vdash (\lambda x\ .\ s)\ t \colon \psi}
\quad \longrightarrow \quad
\begin{array}{c}\vdots \\ s[t/x] \\ \vdots \\ \Gamma \vdash s[t/x] \colon \psi\end{array}
$$

Compile-time reduction rules

$$
\begin{array}{lll}
\mathsf{fst}(\mathsf{cases}(s, t, r)) & \longrightarrow_r & \mathsf{cases}(s, \mathsf{fst} \circ t, \mathsf{fst} \circ r) \\
\mathsf{scd}(\mathsf{cases}(s, t, r)) & \longrightarrow_r & \mathsf{cases}(s, \mathsf{scd} \circ t, \mathsf{scd} \circ r) \\
\mathsf{cases}(\mathsf{cases}(s, t, r), u, w) & \longrightarrow_r & \mathsf{cases}(s, \lambda v\ .\ \mathsf{cases}(t(v), u, w), \\
 & & \qquad \lambda v\ .\ \mathsf{cases}(r(v), u, w)) \\
(u\ (\mathsf{cases}(s, t, r))) & \longrightarrow_r & \mathsf{cases}(s, u \circ t, u \circ r)
\end{array}
$$

Reducing a cases/elim pair: $\wedge$

$$
\dfrac{\dfrac{\Gamma \vdash s \colon \varphi \vee \psi \qquad \Gamma, x \colon \varphi \vdash t \colon \chi_1 \wedge \chi_2 \qquad \Gamma, y \colon \psi \vdash r \colon \chi_1 \wedge \chi_2}{\Gamma \vdash \mathsf{cases}(s, \lambda x\ .\ t, \lambda y\ .\ r) \colon \chi_1 \wedge \chi_2}}{\Gamma \vdash \mathsf{fst}(\mathsf{cases}(s, \lambda x\ .\ t, \lambda y\ .\ r)) \colon \chi_1}
$$

$$\longrightarrow$$

$$
\dfrac{\Gamma \vdash s \colon \varphi \vee \psi \qquad \dfrac{\Gamma, x \colon \varphi \vdash t \colon \chi_1 \wedge \chi_2}{\Gamma, x \colon \varphi \vdash \mathsf{fst}(t) \colon \chi_1} \qquad \dfrac{\Gamma, y \colon \psi \vdash r \colon \chi_1 \wedge \chi_2}{\Gamma, y \colon \varphi \vdash \mathsf{fst}(r) \colon \chi_1}}{\Gamma \vdash \mathsf{cases}(s, \lambda x\ .\ \mathsf{fst}(t), \lambda y\ .\ \mathsf{fst}(r)) \colon \chi_1}
$$

Reducing a cases/elim pair: $\vee$

Omitted: Analogous deriv. of $\Gamma, y \colon \psi \vdash g_2 \colon \rho$

$$
\dfrac{\dfrac{\Gamma \vdash s \colon \varphi \vee \psi \qquad \begin{array}{c}\Gamma, x \colon \varphi \vdash t \colon \chi_1 \vee \chi_2 \\ \Gamma, y \colon \psi \vdash r \colon \chi_1 \vee \chi_2\end{array}}{\Gamma \vdash \mathsf{cases}(s, \lambda x\ .\ t, \lambda y\ .\ r) \colon \chi_1 \vee \chi_2} \qquad \begin{array}{c}\Gamma, z \colon \chi_1 \vdash u \colon \rho \\ \Gamma, v \colon \chi_2 \vdash w \colon \rho\end{array}}{\Gamma \vdash f \colon \rho}
$$

$$\longrightarrow$$

$$
\dfrac{\Gamma \vdash s \colon \varphi \vee \psi \qquad \dfrac{\Gamma, x \colon \varphi \vdash t \colon \chi_1 \vee \chi_2 \qquad \begin{array}{c}\Gamma, x \colon \varphi, z \colon \chi_1 \vdash u \colon \rho \\ \Gamma, x \colon \varphi, v \colon \chi_2 \vdash w \colon \rho\end{array}}{\Gamma, x \colon \varphi \vdash g_1 \colon \rho}}{\Gamma \vdash f' \colon \rho}
$$

$$f = \mathsf{cases}(\mathsf{cases}(s, \lambda x\ .\ t, \lambda y\ .\ r), \lambda z\ .\ u, \lambda v\ .\ w)$$

$$g_1 = \mathsf{cases}(t, \lambda z\ .\ u, \lambda v\ .\ w) \quad g_2 = \mathsf{cases}(r, \lambda z\ .\ u, \lambda v\ .\ w)$$

$$f' = \mathsf{cases}(u, g_1, g_2)$$

The Reduction Relation

$$\frac{s \longrightarrow_r t}{s \longrightarrow t} \qquad \frac{s \longrightarrow t}{\mathcal{C}[s] \longrightarrow \mathcal{C}[t]}$$

$$\frac{}{s \longrightarrow^* s} \qquad \frac{s \longrightarrow^* t \quad t \longrightarrow u}{s \longrightarrow^* u}$$

Contexts $\mathcal{C}[x]$

Replace any s, t, u with an x to make a context $\mathcal{C}[x]$:

$$\begin{array}{rclclcl} \mathcal{C}[x] & ::= & x & \mid & & & \\ & & \langle \mathcal{C}'[x], t\rangle & \mid & \langle s, \mathcal{C}'[x]\rangle & \mid & \\ & & \mathrm{fst}(\mathcal{C}'[x]) & \mid & \mathrm{scd}(\mathcal{C}'[x]) & \mid & \\ & & (\lambda v\ .\ \mathcal{C}'[x]) & \mid & (\mathcal{C}'[x]\ t) & \mid & (s\ \mathcal{C}'[x]) \\ & & \langle \mathrm{lft}, \mathcal{C}'[x]\rangle & \mid & \langle \mathrm{rgt}, \mathcal{C}'[x]\rangle & \mid & \\ & & \mathrm{cases}(\mathcal{C}'[x], t, r) & \mid & \mathrm{cases}(s, \mathcal{C}'[x], r) & \mid & \mathrm{cases}(s, t, \mathcal{C}'[x]) \end{array}$$

Two Computational Theorems

Type Preservation and Normalization

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Theorem (Normal Form)

If $\Gamma \vdash s\colon \varphi$, then there is a normal form t such that $s \longrightarrow^* t$.

A Corollary: Normal Derivations

Corollary

1. *If φ is derivable from Γ, then there is a normal derivation t such that $\Gamma \vdash t\colon \varphi$*
2. *If additionally $\Gamma = \emptyset$, then t is closed (i.e. no free variables)*

Proof.

1. If φ is derivable from Γ, then for some s, $\Gamma \vdash s\colon \varphi$.
By normal form, $s \longrightarrow t$ for some normal t.
By type preservation, $\Gamma \vdash t\colon \varphi$.

2. Via the Context Lemma, which says:

If $\Gamma \vdash s\colon \varphi$, then $\mathrm{fv}(s) \subseteq \mathrm{dom}(\Gamma)$. □

A Normal Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p,\ (p\to\bot)\wedge q \vdash (p\to\bot)\wedge q}{p,\ (p\to\bot)\wedge q \vdash p\to\bot} \qquad p,\ (p\to\bot)\wedge q \vdash p}{p,\ (p\to\bot)\wedge q \vdash \bot}}{p,\ (p\to\bot)\wedge q \vdash q}}{(p\to\bot)\wedge q \vdash p\to q}}{\vdash ((p\to\bot)\wedge q)\to(p\to q)}$$

$$\lambda x\ .\ \lambda y\ .\ \mathrm{emp}(\mathrm{fst}(x)\ y)$$

Another Normal Proof

$$\dfrac{\dfrac{\dfrac{p,(p\vee q)\to r \vdash (p\vee q)\to r \qquad \dfrac{p,(p\vee q)\to r \vdash p}{p,(p\vee q)\to r \vdash p\vee q}}{p,(p\vee q)\to r \vdash r}}{(p\vee q)\to r \vdash p\to r}}{\vdash ((p\vee q)\to r)\to(p\to r)}$$

$$\lambda y\ .\ \lambda x\ .\ (y\ \langle \mathrm{fst}, x\rangle)$$

Normal Derivations, 1

Regarding s as a tree with the conclusion at the root

If d is a normal derivation, then

working upward from any point through *major premises*,
every application of an introduction rule
is reached before
any application of an elimination rule

All premises in an introduction rule are major

The major premise of an elimination rule is
the premise containing the connective to be eliminated

Major and Minor Premises: Conjunction

$$\frac{\Gamma \vdash s\colon \varphi \qquad \Gamma \vdash t\colon \psi}{\Gamma \vdash \langle s, t\rangle\colon \varphi \wedge \psi}$$

$$\frac{\Gamma \vdash s\colon \varphi \wedge \psi}{\Gamma \vdash \mathrm{fst}(s)\colon \varphi} \qquad \frac{\Gamma \vdash s\colon \varphi \wedge \psi}{\Gamma \vdash \mathrm{scd}(s)\colon \psi}$$

Major and Minor Premises: Implication

$$\frac{\Gamma, x\colon \varphi \vdash s\colon \psi}{\Gamma \vdash \lambda x\,.\, s\colon \varphi \to \psi}$$

$$\frac{\Gamma \vdash s\colon \varphi \to \psi \qquad \Gamma \vdash t\colon \varphi}{\Gamma \vdash (s\ t)\colon \psi}$$

Major and Minor Premises: Disjunction

$$\frac{\Gamma \vdash s\colon \varphi}{\Gamma \vdash \langle \mathrm{lft}, s\rangle\colon \varphi \vee \psi} \qquad \frac{\Gamma \vdash s\colon \psi}{\Gamma \vdash \langle \mathrm{rgt}, s\rangle\colon \varphi \vee \psi}$$

$$\frac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \Gamma, x\colon \varphi \vdash t\colon \chi \qquad \Gamma, y\colon \psi \vdash r\colon \chi}{\Gamma \vdash \mathrm{cases}(s, \lambda x\,.\, t, \lambda y\,.\, r)\colon \chi}$$

Major and Minor Premises: Axiom and Falsehood

$$\frac{}{\Gamma, x\colon \varphi \vdash x\colon \varphi} \qquad \frac{\Gamma \vdash x\colon \bot}{\Gamma \vdash \mathrm{emp}(x)\colon \varphi}$$

Normal Derivations, 1

Regarding s as a tree with the conclusion at the root

If d is a normal derivation, then

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every application of an introduction rule
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All premises in an introduction rule are major

The major premise of an elimination rule is
the premise containing the connective to be eliminated

Normal Derivations, 2

If d is a normal derivation, and p is any upwards path in d

- **if** p traverses only elimination rules
- **and** p traverses a disjunction elimination inference
- **then** it is below any other elimination rule

By the compile-time rules

Reducing a cases/elim pair: ∧

$$\dfrac{\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \Gamma, x\colon \varphi \vdash t\colon \chi_1 \wedge \chi_2 \qquad \Gamma, y\colon \psi \vdash r\colon \chi_1 \wedge \chi_2}{\Gamma \vdash \mathsf{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r)\colon \chi_1 \wedge \chi_2}}{\Gamma \vdash \mathsf{fst}(\mathsf{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r))\colon \chi_1}$$

$$\longrightarrow$$

$$\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \dfrac{\Gamma, x\colon \varphi \vdash t\colon \chi_1 \wedge \chi_2}{\Gamma, x\colon \varphi \vdash \mathsf{fst}(t)\colon \chi_1} \qquad \dfrac{\Gamma, y\colon \psi \vdash r\colon \chi_1 \wedge \chi_2}{\Gamma, y\colon \varphi \vdash \mathsf{fst}(r)\colon \chi_1}}{\Gamma \vdash \mathsf{cases}(s, \lambda x\,.\,\mathsf{fst}(t), \lambda y\,.\,\mathsf{fst}(r))\colon \chi_1}$$

Normal Derivations, 3

If d is a normal derivation, and p is any upwards path in d

- **if** p traverses only elimination rules
- **then** p traverses at most one major premise of a disjunction elimination inference

By the case/elim rule for ∨
and the previous claim

Reducing a cases/elim pair: ∨

Omitted: Analogous deriv. of $\Gamma, y\colon \psi \vdash g_2\colon \rho$

$$\dfrac{\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \begin{matrix}\Gamma, x\colon \varphi \vdash t\colon \chi_1 \vee \chi_2 \\ \Gamma, y\colon \psi \vdash r\colon \chi_1 \vee \chi_2\end{matrix}}{\Gamma \vdash \mathsf{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r)\colon \chi_1 \vee \chi_2} \qquad \begin{matrix}\Gamma, z\colon \chi_1 \vdash u\colon \rho \\ \Gamma, v\colon \chi_2 \vdash w\colon \rho\end{matrix}}{\Gamma \vdash f\colon \rho}$$

$$\longrightarrow$$

$$\dfrac{\Gamma \vdash s\colon \varphi \vee \psi \qquad \dfrac{\Gamma, x\colon \varphi \vdash t\colon \chi_1 \vee \chi_2 \qquad \begin{matrix}\Gamma, x\colon \varphi, z\colon \chi_1 \vdash u\colon \rho \\ \Gamma, x\colon \varphi, v\colon \chi_2 \vdash w\colon \rho\end{matrix}}{\Gamma, x\colon \varphi \vdash g_1\colon \rho}}{\Gamma \vdash f'\colon \rho}$$

$$f = \mathsf{cases}(\mathsf{cases}(s, \lambda x\,.\,t, \lambda y\,.\,r), \lambda z\,.\,u, \lambda v\,.\,w)$$

$$g_1 = \mathsf{cases}(t, \lambda z\,.\,u, \lambda v\,.\,w) \qquad g_2 = \mathsf{cases}(r, \lambda z\,.\,u, \lambda v\,.\,w)$$

$$f' = \mathsf{cases}(u, g_1, g_2)$$

Normal Derivations, 4

If d is a normal derivation, and p is any upwards path in d

- **if** p traverses only introduction rules
- **then** each successive right hand side is a subformula of the one below it

By the form of the introduction rules

Today's Goal

Theorem

Every normal derivation has the subformula property

Why it's true

Theorem

Every normal derivation d of $\Gamma \vdash s : \varphi$ has the subformula property

Proof.

1. Conclusion of an introduction rule: subformula of φ
2. Major premise of an elimination rule: subformula of some ψ in Γ
3. Minor premise of $\rightarrow$-elimination: subformula of the major premise
4. Minor premise of $\vee$-elimination: subformula of φ

$\square$

Today's Goal

Theorem

Every normal derivation has the subformula property

Corollary

The derivation rules are consistent:
there is no derivation $\vdash s : \bot$