CS 543:
Computer Graphics

Points, Scalars, Vectors

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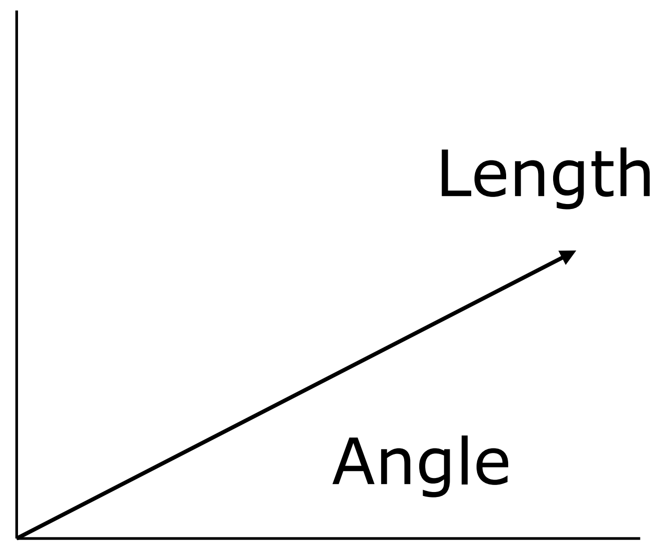
(with lots of help from Prof. Emmanuel Agu :-)

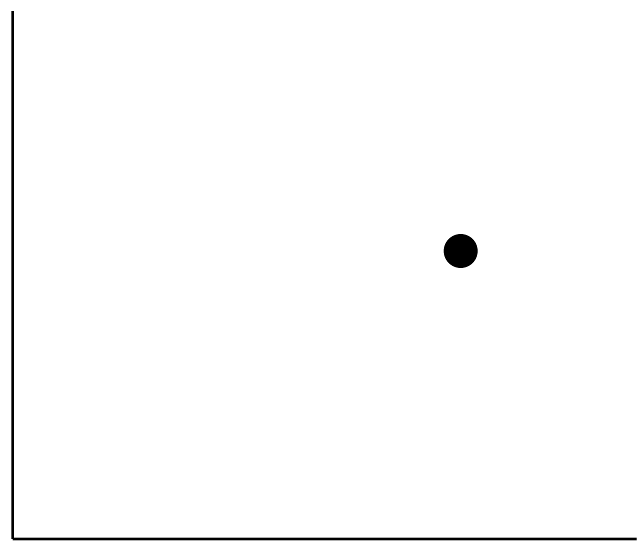
Points and Vectors

- Points, vectors defined relative to a coordinate system
- Vectors
  - Magnitude
  - Direction
  - **NO** position
- Can be
  - added, scaled, rotated
- CG vectors
  - 2, 3 or 4 dimensions



Points

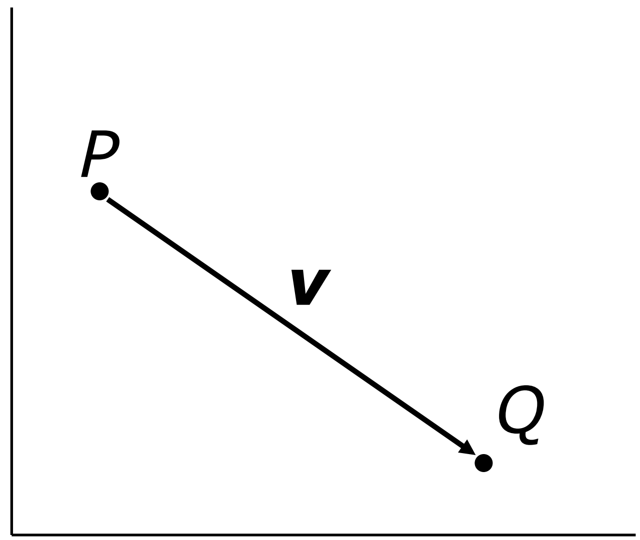
- Location in coordinate system
- Cannot add or scale
- Subtract 2 points = vector



Vector-Point Relationship

- Difference between 2 points = vector

$$\mathbf{v} = Q - P$$



- Sum of point and vector = point

$$P + \mathbf{v} = Q$$

Vector Operations

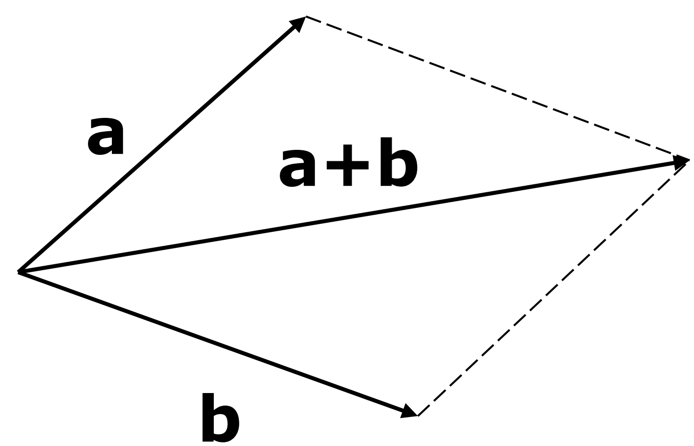
- Define vectors

$$\mathbf{a} = (a_1, a_2, a_3)$$

$$\mathbf{b} = (b_1, b_2, b_3)$$

- Then, vector addition

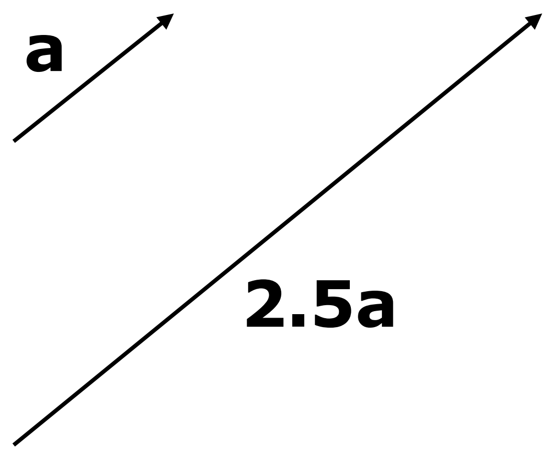
$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$



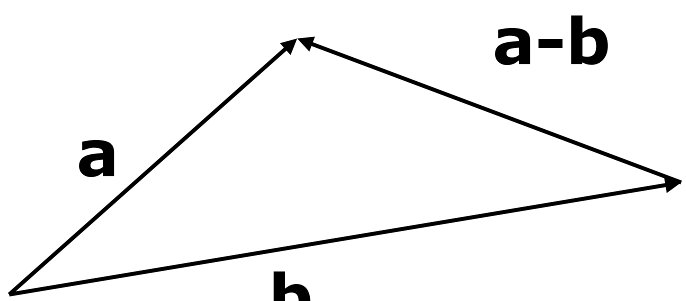
Vector Operations (cont.)

- Scaling a vector by a scalar, *s*
  - This is *uniform* scaling

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s)$$



- Vector subtraction



$$\mathbf{a} - \mathbf{b}$$

$$= (a_1 + (-b_1), a_2 + (-b_2), a_3 + (-b_3))$$

Vector Operation Examples

- Scaling a vector by a scalar

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s)$$

- Vector addition

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

- Examples
  - Assume: *a*=(2, 5, 6), *b*=(-2, 7, 1), *s*=6

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) = (0,12,7)$$

$$\mathbf{a}s = (a_1 s, a_2 s, a_3 s) = (12,30,36)$$

Magnitude of a Vector

- Magnitude of *a*

$$|\mathbf{w}| = \sqrt{w_1{}^2 + w_2{}^2 + ... + w_n{}^2}$$

- Normalizing a vector (*unit vector*)

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{vector}{magnitude}$$

- Note: Magnitude of a normalized vector is 1, *i.e.*,

$$\sqrt{w_1{}^2 + w_2{}^2 + ... + w_m{}^2} = 1$$

Dot (Scalar) Product

- Dot product

$$d = \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

- Result is a number
- For example, if *a*=(2,3,1) and *b*=(0,4,-1)

$$\mathbf{a} \cdot \mathbf{b} = 2 * 0 + 3 * 4 + 1 * -1$$

$$= 0 + 12 - 1 = 11$$

Properties of Dot Products

- Symmetry (or commutative):

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

- Linearity:

$$(\mathbf{a} + \mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$$

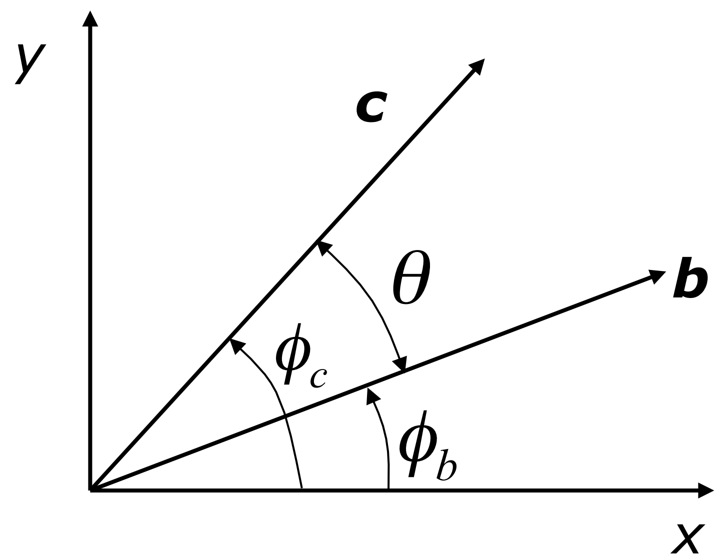
- Homogeneity:

$$(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$$

- And

$$\left|\mathbf{b}^2\right| = \mathbf{b} \cdot \mathbf{b}$$

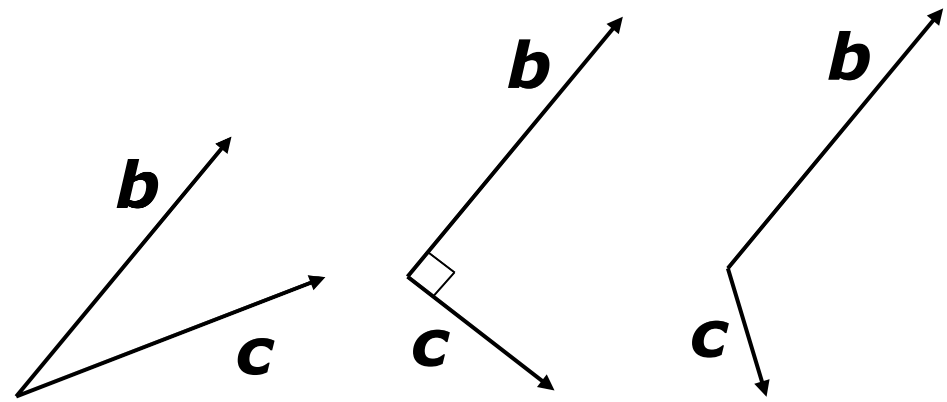
Angle Between Two Vectors



$$\mathbf{b} = \left(|\mathbf{b}|\cos\phi_b, |\mathbf{b}|\sin\phi_b\right)$$

$$\mathbf{c} = \left(|\mathbf{c}|\cos\phi_c, |\mathbf{c}|\sin\phi_c\right)$$

$$\mathbf{b}\cdot\mathbf{c} = |\mathbf{b}||\mathbf{c}|\cos\theta$$



b.c > 0 b.c = 0 b.c < 0

Sign of **b.c** tells us something about the angle.

Angle Between Two Vectors (cont.)

- Find the angle between the vectors

  b = (3, 4) and ***c* =** (5, 2)

  |**b**|= 5, |**c**|= 5.385

$$\hat{\mathbf{b}} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\hat{\mathbf{c}} = (.9285, .3714)$$

$$\hat{\mathbf{b}} \bullet \hat{\mathbf{c}} = 0.85422 = \cos\theta$$

$$\theta = 31.326^\circ$$

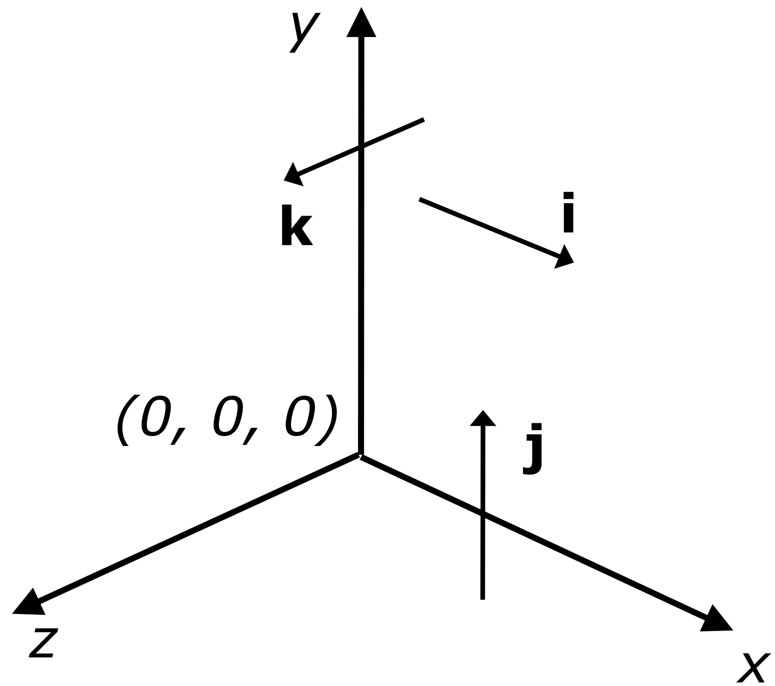
Standard Unit Vectors

- Define

$$\mathbf{i} = (1,0,0)$$

$$\mathbf{j} = (0,1,0)$$

$$\mathbf{k} = (0,0,1)$$



- So that any vector

$$\mathbf{v} = (a,b,c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Cross (Vector) Product

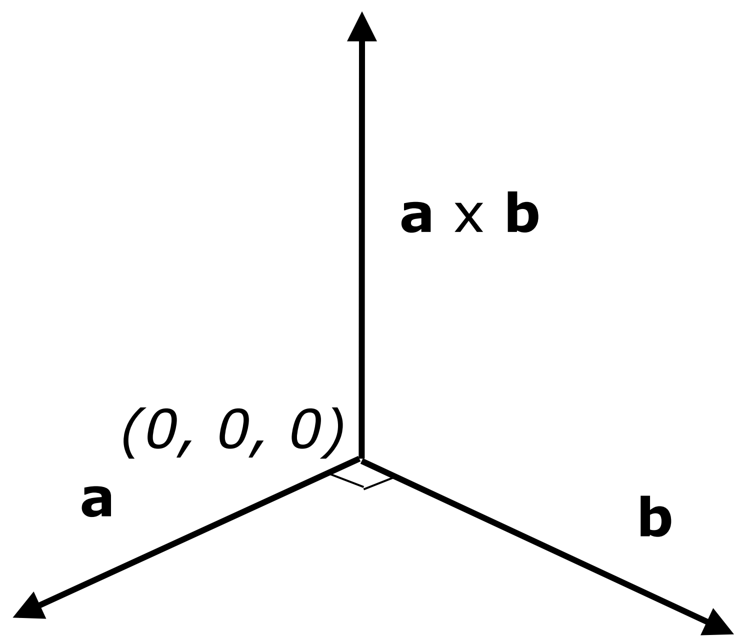
- If $\mathbf{a} = (a_x, a_y, a_z)$ $\mathbf{b} = (b_x, b_y, b_z)$
- Then $\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$
- Remember using determinant

$$\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

- Note: **a** x **b** is perpendicular to **a** and **b**

Cross (Vector) Product (cont.)

- Note: **a** x **b** is perpendicular to **a** and **b**



Cross (Vector) Product (cont.)

- Calculate **a x b** if a = (3, 0, 2) and **b** = (4, 1, 8)

a x b = -2**i** – 16**j** + 3**k**

Recall: $\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$