

## Seven Rules for Big- $O$ and $\Theta^*$

Here are seven rules that you can use to solve problems involving big- $O$  and  $\Theta$ . They will solve the big majority of the big- $O$  and  $\Theta$  comparisons you'll need in this course (and for a long way beyond). Two assumptions are noted in the *Fine Print* on the back.

$$\Theta(c \cdot f(x)) = \Theta(f(x)) \quad (1)$$

$$\Theta(f(x) + g(x)) = \Theta(\max(f(x), g(x))) \quad (2)$$

$$\Theta(f(x) \cdot h(x)) \leq \Theta(g(x) \cdot h(x)) \quad \text{if and only if} \quad \Theta(f(x)) \leq \Theta(g(x)) \quad (3)$$

$$\Theta(x^c) \leq \Theta(x^d) \quad \text{if and only if} \quad c \leq d \quad (4)$$

$$\Theta(\log x) < \Theta(x^c) \quad \text{if and only if} \quad 0 < c \quad (5)$$

Assuming that  $c > 0$ ,

$$\Theta(x^c) < \Theta(d^x) \quad \text{if and only if} \quad 1 < d \quad (6)$$

Assuming that  $1 \leq c$  and  $1 \leq d$ ,

$$\Theta(c^x) < \Theta(d^x) \quad \text{if and only if} \quad c < d \quad (7)$$

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**Fine Print.**  $\Theta(f)$  means the set of all functions  $g$  that grow essentially as fast as  $f$ . Officially,  $\Theta(f) =$

$$\{g: \text{there exist } N_0, c_1, c_2 \text{ such that, for all } x > N_0, \\ g(x) \leq c_1 \cdot f(x) \text{ and } f(x) \leq c_2 \cdot g(x)\}.$$

So  $\Theta(f) = \Theta(g)$ ,  $g \in \Theta(f)$ , and  $f \in \Theta(g)$  all mean the same thing.

Big- $O$  makes an ordering on the  $\Theta$ -classes. By  $\Theta(f) \leq \Theta(g)$ , we mean that  $f \in O(g)$ . In fact, when  $f \in O(g)$ , either  $f \in \Theta(g)$ , or else every function  $g' \in \Theta(g)$  asymptotically dominates every function  $f' \in \Theta(f)$ . So this ordering works in a compatible way across whole  $\Theta$ -classes.

$\Theta(f) < \Theta(g)$  means  $f \in O(g)$  but  $f \notin \Theta(g)$ .

A function  $f$  is *non-decreasing* if  $x \leq y$  implies  $f(x) \leq f(y)$ . It's *eventually non-decreasing* if  $N_0 < x \leq y$  implies  $f(x) \leq f(y)$ , for some  $N_0$ . A function is *eventually positive* if, for some  $N_0$ , for all  $x > N_0$ ,  $f(x) > 0$ .

In the rules above, assume all the functions  $f, g$  are:

- eventually non-decreasing, and
- eventually positive.